

# Hamiltonian Theories of the FQHE

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This paper reviews progress on the Fractional Quantum Hall Effect (FQHE) based on what we term hamiltonian theories, i.e., theories that proceed from the microscopic electronic hamiltonian to the final solution via a sequence of transformations and approximations, either in the hamiltonian or path integral approach, as compared to theories based on exact diagonalization or trial wavefunctions. We focus on the Chern-Simons (CS) approach in which electrons are converted to CS fermions or bosons that carry along flux tubes and our Extended Hamiltonian Theory (EHT) in which electrons are paired with pseudo-vortices to form composite fermions (CF) whose properties are a lot closer to the ultimate low-energy quasiparticles. We address a variety of qualitative and quantitative questions: In what sense do electrons really bind to vortices? What is the internal structure of the Composite Fermion and what does it mean? What exactly is the dipole picture? What degree of freedom carries the Hall current when the quasiparticles are localized or neutral or both? How exactly is the kinetic energy quenched in the lowest Landau level and resurrected by interactions? How well does the CF picture work at and near  $\nu = 1/2$ ? Is the system compressible at  $\nu = 1/2$ ? If so, how can Composite Fermions be dipolar at  $\nu = 1/2$  and still be compressible? How is compressibility demonstrated experimentally? How does the charge of the excitation get renormalized from that of the electron to that of the CF in an operator treatment? Why do Composite Fermions sometimes appear to be free when they are not? How are (approximate) transport gaps, zero-temperature magnetic transitions, the temperature dependent polarizations of gapped and gapless states, the NMR relaxation rate  $1/T_1$  in gapless states, and gaps in inhomogeneous states computed? It is seen that though the CS and EHT approaches agree whenever a comparison is possible, results that are transparent in one approach are typically opaque in the other, making them truly complementary.

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## I. INTRODUCTION

Twenty years ago, von Klitzing, Dorda, and Pepper (1980) made the first discovery in what has proved to be a vital and exciting subfield of condensed matter physics to the present day, that of the quantum Hall effects (QHE). Their discovery that the Hall conductance  $\sigma_{xy}$  of a two-dimensional electron gas is quantized at integer multiples of  $e^2/2\pi\hbar$ , the quantum unit of conductance, is known as the integer quantum Hall effect (IQHE). Soon after, Tsui, Störmer, and Gossard (1982) discovered the even more puzzling fractional quantum Hall effect (FQHE). Since the first observation of the fraction  $\frac{1}{3}$ , many more have been seen (see, for example, Störmer, Tsui, and Gossard 1999).

This review focuses on what we term the *hamiltonian theories* of the FQHE, by which we mean theoretical approaches that begin with the microscopic hamiltonian for interacting electrons and try to obtain a satisfactory description of the underlying physics through a sequence of transformations and approximations in the operator or path integral formalism. We bother to give a special name to what appears to be business as usual, because an alternate

approach, pioneered by Laughlin (1983a,b) and based on writing trial wavefunctions, has proven so extraordinarily successful. While we will of course discuss the wavefunction approach here because the hamiltonian approach is inspired by it, the discussion (of this and any other topic) will be aimed at serving our primary goal: Providing a cogent and critical description of the hamiltonian approach in one place, including all the hindsight and insight the intervening years have provided. Of necessity, many topics will have to be omitted or treated summarily. Fortunately many excellent reviews exist<sup>1</sup> and the reader is directed to them.

The hamiltonian theories described here address both qualitative and quantitative issues. They give a concrete operator realization of many heuristic pictures that have been espoused and make precise under what conditions and in what sense these pictures are valid. They allow one to compute (within reasonable approximations) a large number of quantities at zero and nonzero temperatures at equal and unequal times. They open the door for a treatment of disorder.

The hamiltonian theories themselves fall into two categories. The first, which we call the Chern-Simons (CS) approach, consists of making a singular gauge transformation on the electronic wavefunction that leads to a composite particle which is the union of an electron and some number of point flux tubes. The composite particles have a nondegenerate ground state at mean-field level. They are coupled to a gauge field that is fully determined by the particle coordinates. This formalism is best suited for computation of a variety of response functions. It is particularly effective at  $\nu = 1/2$ , where adherence to gauge invariance is of paramount importance if certain low-energy phenomena pertaining to the overdamped mode, coupling to surface acoustic waves, or the compressibility are to be properly described. On the other hand, the composite particles do not exhibit in any transparent way the quasiparticle properties (such as charge  $e^*$  or effective mass  $m^*$ ) deduced from trial wavefunctions, though in principle they would surface after considerable work. It is also very hard to obtain a smooth limit as the electron mass  $m \rightarrow 0$  in this approach. The Extended Hamiltonian Theory (EHT), which we developed over the years addresses some of these issues.<sup>2</sup> By the adjective *extended* we signify both that our work is an extension of older CS work and that our hamiltonian is defined in an extended or enlarged Hilbert space with additional degrees of freedom. The enlarged space allows us to introduce a quasiparticle that is a much better approximation to what we expect, to easily disentangle low-energy (Lowest Landau Level) and high-energy physics, to compute a variety of gaps, finite temperature properties, to describe inhomogeneous states and so on. On the other hand, the proper treatment of the additional degrees of freedom is very difficult to ensure in the computation of certain very low-energy quantities. Both the nonzero compressibility and an overdamped mode at  $\nu = 1/2$  (and its experimental consequences) are all but invisible in this approach, though they can be extracted with some effort. No contradictions exist between the predictions of the CS and EHT approaches, which by and large tend to make predictions in complementary regions. Their predictions can be shown to agree in overlapping regions, but only with some effort.

## A. The experiment

Figure 1 shows in schematic form the experiment in a rectangular geometry, with the current  $j_x$ , magnetic field  $B_z$  and electric field  $E_y$  in mutually perpendicular directions. By definition

$$j_x = \sigma_{xy} E_y. \quad (1)$$

If we multiply both sides by the width of the sample we obtain

$$I_x = \sigma_{xy} V_y \quad (2)$$

as the relation between current and voltage, reminding us that in this  $d = 2$  problem the Hall conductance and conductivity are the same.<sup>3</sup>

Let us ask what one expects for  $\sigma_{xy}$  based on the simplest ideas, so as to place the experimental discoveries in perspective. Let us ignore interactions and disorder. Then we can assert that in the steady state, the electric and magnetic forces balance:

$$eE = evB \quad (3)$$

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<sup>1</sup> The earliest comprehensive introduction to the full range of the quantum Hall effects is Prange and Girvin (1987). Huckestein (1995) concentrates on the integer quantum Hall effect. A more recent comprehensive review is Das Sarma and Pinczuk (1997), while a more focused treatment of the enormous body of research on Composite Fermions is summarized in Heinonen (1998).

<sup>2</sup> We used to refer to this as THE hamiltonian formalism, but were persuaded to consider a name change that better reflected the state of affairs. Our compliance could not have been more total- the new acronym is the exact reverse of the old one.

<sup>3</sup> The longitudinal conductance and conductivity are related by the (aspect) ratio of length to width.

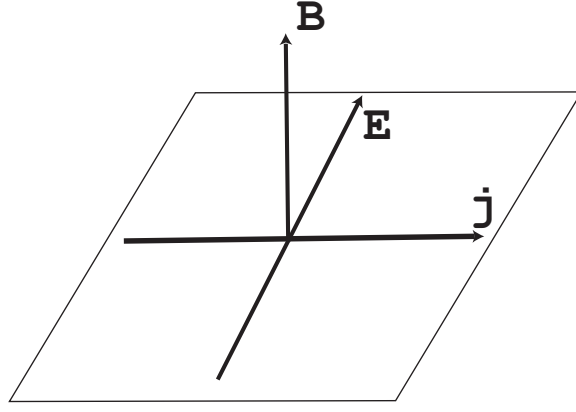


FIG. 1 The Hall experiment in a rectangular geometry.

where  $e$  is the charge of the electrons, and  $v$  is their velocity. (Vector indices are omitted when obvious.) Therefore

$$j = nev = \frac{neE}{B} \quad (4)$$

where  $n$  is the number density. Thus

$$\sigma_{xy} = \frac{ne}{B}. \quad (5)$$

This result is unaffected by interactions and relies only on Galilean invariance (relativity). To see this, let us perform a boost to a frame in which the electric field (to leading order in  $v/c$ )

$$\mathbf{E}' = \mathbf{E} - \mathbf{v} \times \mathbf{B} \quad (6)$$

vanishes, that is, to a frame with  $v = E/B$ . In this frame  $j = 0$ . Boosting back to the lab frame we obtain  $j = neE/B$  and regain Eqn.(5).

Figure 2 is a caricature of what is measured at  $T = 0$  in the thermodynamic limit. The key feature is that  $\sigma_{xy}$ , instead of varying linearly with  $ne/B$ , is quantized at steps or plateaus. This article focuses on steps given by

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar} \nu, \quad (7)$$

where

$$\nu = \frac{p}{2ps + 1} \quad p = 1, 2, \quad s = 0, 1, 2, \dots \quad (8)$$

The IQHE corresponds to  $s = 0$ ,  $\nu = p$ , while FQHE refers to  $s > 0$ .<sup>4</sup> We explicitly display the  $\hbar$  which will soon be set equal to unity.

On each step the longitudinal conductances vanish. Thus the conductivity tensor assumes the form

$$\sigma_{ij} = \frac{e^2}{2\pi\hbar} \begin{pmatrix} 0 & \nu \\ -\nu & 0 \end{pmatrix}. \quad (9)$$

Consequently the diagonal part of the resistivity tensor is also zero and the transport is dissipationless. In the real world, due to nonzero temperature  $T$  and finite sample size, the transitions between steps acquire a finite width, with  $\sigma_{xx} > 0$  therein.

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<sup>4</sup> Fractions with denominator  $2ps - 1$  require only minor modifications and are left as an exercise.

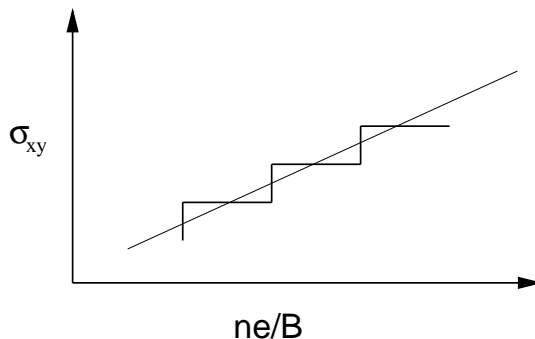


FIG. 2 Schematic form of measured Hall conductance at  $T = 0$ . The straight line is the Galilean invariant result, while the steps describe the data.

Why is the quantization of conductance so surprising, given that we have seen so many instances of quantization of observables before? The answer is that the observable in this case does not refer to an atom or molecule, but to a *macroscopic* sample, with sample specific disorder. The role of disorder is truly paradoxical in the QHE: on the one hand, without disorder, we cannot escape the Galilean invariant relation  $\sigma_{xy} = ne/B$ ; on the other hand, despite disorder (which surely varies from sample to sample), the conductance in any step is constant to better than a few parts in  $10^{10}$ ! A fairly detailed explanation of this now exists and will be described soon. An integral part of this resolution is also why  $\sigma_{xx} = 0$ , for only then can we understand why  $\sigma_{xy}$  is so well quantized even though the leads used to measure  $V_y$  may not be perfectly aligned—there will be no unwanted contributions from the longitudinal voltage drop.

### B. What is special about the steps?

In searching for the physics underlying any one step or plateau, it makes sense to begin with the points where the straight line (the Galilean invariant result) intercepts the steps. Not only are these points singled out by experiment, but they also have a Hall conductance that we could hope to understand without including disorder.<sup>5</sup>

At these points we have

$$\frac{ne}{B} = \frac{e^2}{2\pi\hbar} \frac{p}{2ps + 1} \quad (10)$$

which we rewrite as

$$\frac{B}{n(2\pi\hbar/e)} = 2s + \frac{1}{p}. \quad (11)$$

Thus the ratio of  $B$ , the flux density, and  $n$ , the particle density, has some special rational values at these points. For example in the case of the fraction  $1/3$  there are three quanta of flux per electron.

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<sup>5</sup> In the absence of disorder any state would have the desired Hall conductance. We are looking for a nontrivial correlated state that is robust under the inclusion of disorder and capable of dominating the plateau it belongs to.

To understand what is so special about these values, recall the following textbook results (Shankar 1994) about a single particle of mass  $m$  and charge  $e$  moving in two dimensions in a perpendicular magnetic field.

- The energy is quantized into Landau Levels (LL's) located at  $E = (e\hbar B/m)(n + \frac{1}{2}) \equiv \hbar\omega_0(n + \frac{1}{2})$ , where  $\omega_0$  is called the *cyclotron frequency*.
- Each LL has a degeneracy equal to  $\Phi/\Phi_0$ , the flux in units of the flux quantum  $\Phi_0 = 2\pi\hbar/e$ , or :

$$\text{Degeneracy per unit area of each LL} = \frac{B}{(2\pi\hbar/e)}. \quad (12)$$

- The wavefunctions of the Lowest Landau Level (LLL) are, in the symmetric gauge,

$$\psi_{LLL} = z^m e^{-|z|^2/4l^2} \quad z = x + iy \quad m = 0, 1, \dots \quad (13)$$

where  $l = \sqrt{\hbar/eB}$  is the magnetic length. We will often drop the gaussian factor in LLL wavefunctions. Higher LL wavefunctions depend on both  $z$  and  $\bar{z}$ .

Thus Eqns. (11) and Eqn. (12) imply that at the special points

$$\frac{\text{Number of states in a LL}}{\text{Number of particles}} = \frac{\text{Flux quanta}}{\text{particles}} = \frac{B}{n(2\pi\hbar/e)} = 2s + \frac{1}{p} = \nu^{-1}. \quad (14)$$

Note that  $\nu$ , which stood for the dimensionless conductance, is thus the number of occupied LL's. It is called the *filling factor*. In the FQHE  $\nu$  is a fraction, restricted in this article to be  $< \frac{1}{2}$  unless otherwise mentioned.

If  $\nu < 1$ , there are more LLL states than particles, and in the noninteracting limit they can all be fit into the LLL with room to spare. For example at  $\nu = \frac{1}{3}$ , there are 3 LLL states per electron. *This macroscopic degeneracy in the noninteracting ground state means we cannot even get started with a perturbative treatment of disorder and interactions.* This is the central problem. While this problem exists for any  $\nu$  that is not an integer, experiments suggest that  $\nu^{-1} = 2s + \frac{1}{p}$  is somehow preferred by nature. Indeed at such points there is a way out, but it is not simple perturbation theory. This approach is what this article is all about.

An alternative to perturbation theory, the Hartree-Fock (HF) approximation, also does not work, if applied directly to the electronic hamiltonian. Let us see why. In HF, one takes the interaction (quartic in fermion operators) and obtains an expression quadratic in the fermion operators by replacing various bilinears by their ground-state averages, initially taken as free parameters, and dropping some higher-order fluctuations. The quadratic hamiltonian is then solved and the ground state energy is then minimized as a function of the assumed averages. At the minimum, the actual averages will self-consistently come out equal to the assumed averages. Here is a toy model that illustrates the main ideas. The model has just two Fermion operators  $c$  and  $d$ :

$$H = \varepsilon_c c^\dagger c + \varepsilon_d d^\dagger d + u_0 c^\dagger c d^\dagger d \quad (15)$$

Let us separate out the bilinears into averages  $\langle c^\dagger c \rangle = \lambda_c$  and  $\langle d^\dagger d \rangle = \lambda_d$  and fluctuating parts :  $c^\dagger c$  : and :  $d^\dagger d$  :

$$c^\dagger c = : c^\dagger c : + \langle c^\dagger c \rangle \equiv : c^\dagger c : + \lambda_c \quad (16)$$

$$d^\dagger d = : d^\dagger d : + \langle d^\dagger d \rangle \equiv : d^\dagger d : + \lambda_d. \quad (17)$$

We now rewrite  $H$  as

$$H = \varepsilon_c c^\dagger c + \varepsilon_d d^\dagger d + u_0 [\lambda_d c^\dagger c + \lambda_c d^\dagger d - \lambda_c \lambda_d + : c^\dagger c : d^\dagger d :] \quad (18)$$

and neglect the last term, quadratic in the fluctuations, to obtain the HF hamiltonian.

Let  $E_0(\lambda_c, \lambda_d)$  be the energy of the ground state  $|0\rangle$  of the HF hamiltonian. From the Feynman- Hellman theorem

$$\frac{\partial E_0}{\partial \lambda} = \langle 0 | \frac{\partial H}{\partial \lambda} | 0 \rangle \quad (19)$$

which you can prove using the fact that  $\langle 0|0\rangle = 1$  has no  $\lambda$  derivative, it follows that at the minimum,  $\lambda_c = \langle c^\dagger c \rangle$  and  $\lambda_d = \langle d^\dagger d \rangle$ .

Why does this fail in the FQHE problem? For integer filling this procedure gives the expected state of an integer number of LL's filled (at least for not too strong interactions). However, it turns out that for generic fractional filling translational symmetry is spontaneously broken in HF, and the solution corresponds to a crystalline (usually a Wigner Crystal) state. Due to the breaking of translational symmetry, the LL and angular momentum indices  $(n, m)$  are no longer good quantum numbers. The single-particle states are now bands in the crystal, parametrized by a Bloch quasimomentum  $\mathbf{k}$  and a band index  $n_b$ . The HF hamiltonian for FQHE will of the form  $H = \sum_{n_b, \mathbf{k}} \varepsilon_{n_b}(\mathbf{k}) c_{n_b}^\dagger(\mathbf{k}) c_{n_b}(\mathbf{k})$  up to c-number terms (the analogs of  $\lambda_c \lambda_d$ ). The self-consistent solutions were worked out by Yoshioka and Fukuyama (1979), Fukuyama and Platzman (1982), and Yoshioka and Lee (1982). The HF solution is compressible, translationally non-invariant, and has no preference for any particular density. Thus, it does not describe the FQHE phenomenology.

### C. The IQHE - A warm-up

We begin with a brief look at the IQHE case  $s = 0$ , or  $\nu = p$ , which paves the way for Jain's (1989) view of FQHE as the IQHE of entities called Composite Fermions.

In the IQHE, exactly  $p$  LL's are filled in the noninteracting limit. There is exactly one totally antisymmetric ground state which we denote by  $\chi_p$ . The simplest example is  $p = 1$ , with just the LLL filled up. The corresponding wavefunction is

$$\chi_1 = \text{Det} \begin{vmatrix} z_1^0 & z_1^1 & z_1^2 & \dots \\ z_2^0 & z_2^1 & z_2^2 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \cdot \text{Gaussian} = \prod_{i < j} (z_i - z_j) \exp \left[ - \sum_i \frac{|z_i|^2}{4l^2} \right] \quad (20)$$

This nondegenerate ground state is separated from excited states by a gap equal to the cyclotron energy  $\omega_0$ . Though now we have a starting point for perturbation theory, we still need to actually carry out the perturbative calculations and in particular understand why the Hall conductance is *unaffected* by these perturbations and stays at the Galilean invariant value. We also need to understand why the conductance is unchanged as we make small changes in density, i.e., why there are steps.

The explanation of the IQHE, discovered over the years, can be most easily understood in the noninteracting limit. The single-electron problem with certain special types of disorder can be exactly solved (Prange, 1981, Aoki and Ando, 1981, Prange and Joynt, 1982), and the exact solution shows that the Hall current is independent of disorder. Moving beyond this, Trugman (1983) showed in a seminal paper that for generic disorder in very high magnetic fields the electronic states follow the equipotentials of the disorder potential, and each LL gets broadened into a band. The problem has many fruitful analogies with percolation. In particular, there is exactly one energy at the band center at which extended states can exist, while states at all other energies are localized. When the chemical potential  $\mu$  crosses this energy, the Hall conductance changes by an integer. Contrariwise, when  $\mu$  lies in energy range corresponding to localized states, changes of  $\mu$  (changes of filling) produce no change in  $\sigma_{xy}$ , which explains the steps.

From a somewhat different point of view, Laughlin (1981) and Halperin (1982) showed by a gauge argument that to have a quantized Hall conductance in the noninteracting limit all one needs is  $\sigma_{xx} = 0$ , which is assured if the chemical potential lies in the region of localized states. This argument shows that the Hall conductance (in units of the quantum unit of conductance  $e^2/2\pi\hbar$ ) is a topological integer. There is also a field-theoretic analysis of the noninteracting quantum Hall problem in the presence of disorder which yields similar results (Pruisken, 1984, 1985a,b, and Levine, Libby, and Pruisken, 1983, 1984a,b,c).

The IQHE can also be understood in the other limit, where interactions dominate, as follows: In this case if one is not exactly at  $\nu = \text{integer}$ , the excess (or deficit of) particles form a Wigner Crystal due to interactions. First turn off the interactions and impose an external periodic potential of the same period as the Wigner Crystal. In this case Thouless, Kohmoto, Nightingale, and den Nijs (1982) showed that the dimensionless Hall conductance has to be an integer, and that this integer is topological. Later work by Thouless (1983), Niu and Thouless (1984), and Niu, Thouless, Wu (1985) showed that since this integer is topological, it is robust to adiabatically introducing disorder and interactions. As long as the charge gap does not close, and there are no ground state transitions, the Hall conductance will remain unchanged<sup>6</sup>. This allows one to turn off the external periodic potential as the interactions are being turned on. For further details on the IQHE the reader is referred to an excellent review by Huckestein (1995).

### D. The FQHE

We turn now to fractions, such as  $\frac{1}{3}$ . There is nothing special about disordered single-particle states at such a filling: They are expected to be localized except right near the middle of the Landau band (Trugman, 1983). If we ignore disorder and interactions, we are left with the macroscopically degenerate manifold of many-body states. Presumably interactions will select the ground state. How is one to find it?

Here the story branches into two trails. The one blazed by Laughlin (1983a,b) consists of writing down inspired trial wavefunctions; the other is the hamiltonian approach, which starts with an assault on the degeneracy problem.

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<sup>6</sup> By definition, an integer cannot change continuously.

## 1. Laughlin's answer

After some experimentation, Laughlin wrote down the following celebrated trial wavefunction for  $\nu = 1/(2s + 1)$  :

$$\Psi_{\frac{1}{(2s+1)}} = \prod_{j < i} (z_i - z_j)^{2s+1} \exp(-\sum_i |z_i|^2 / 4l^2) \quad (21)$$

which was shown to have nearly unit overlap with the numerical solution for small systems for generic repulsive interactions (Laughlin, 1983a,b, 1987). We will refer to  $\nu = 1/(2s + 1)$  as Laughlin fractions, as per convention.

Of the many remarkable properties of this function, we list those that have the greatest bearing on what follows. First, it is from the LLL and obeys the Pauli principle under particle exchange, since  $2s + 1$  is odd and spin is assumed polarized. Halperin (1983,1984) zeroed in on one of its central features: *it has no wasted zeros*, by which he meant the following. Consider  $\psi(z_1)$ , which is  $\Psi(z_1, \dots, z_N)$  as a function of any one variable, randomly chosen to be  $z_1$ , with all others held fixed.<sup>7</sup> Given that the sample is penetrated by  $N/\nu = N(2s + 1)$  quanta of flux, the phase of  $\psi(z_1)$  has to have an Aharonov-Bohm phase change of  $2\pi N(2s + 1)$  per particle, or  $N(2s + 1)$  zeros given the LLL condition of analyticity. Only  $N$  of these *had* to lie on other electrons by the Pauli principle. But they all *do* lie on other electrons, thereby keeping the electrons away from each other very effectively, producing a low potential energy. (The kinetic energy is of course the same for any function in the LLL.)

By showing that quantum averages in these ground states are precisely statistical averages in a one-component plasma (Baus and Hansen, 1980, Caillol, Levesque, Weis, and Hansen, 1982), Laughlin (1983a,b) showed that the system is an *incompressible fluid*, which means a fluid that abhors density changes. Unlike a Fermi gas which increases (decreases) its density globally when compressed (decompressed), an incompressible fluid is wedded to a certain density and will first show no response to any applied pressure, and then suddenly nucleate a localized region of different density (just the way a Type II superconductor, in which a magnetic field is not welcome, will allow it to enter in quantized units in a region that turns normal).

Laughlin (1983a,b) also provided the wavefunction for a state with such a localized charge deficit. If one imagines inserting a tiny solenoid at a point  $z_0$  and slowly increasing the flux to one quantum, one must, by gauge invariance, return to an eigenstate of  $H$ , and each particle must undergo a  $2\pi$  phase shift as it goes around  $z_0$ .<sup>8</sup> This condition and analyticity point to the ansatz

$$\Psi_{qh} = \prod_i (z_i - z_0) \Psi_{2s+1} \quad (22)$$

This is a *quasihole*. There is a more complicated state with a quasiparticle.

The prefactor is a *vortex* at  $z_0$ . Since vortices play a significant role in the FQHE let us digress to understand them better. Consider first a real flux tube inserted into the sample at a point  $z_0$ . Clearly every electron will "see" this tube at  $z_0$ , i.e.,  $\psi(z_1)$ , the wavefunction, seen as a function of any one coordinate, chosen to be  $z_1$ , will see a  $2\pi$  phase shift as  $z_1$  goes around  $z_0$ . The flux tube clearly has a reality and location of its own, independent of the locations of the electrons. The vortex is an analytic LLL version of the flux tube: the  $2\pi$  phase change is accompanied by a zero, i.e., there is an analytic zero at  $z_0$  for *every coordinate*. The zeros associated with the vortex have the key feature that their location does not depend on the location of the particles, i.e., it too has an independent reality and location like the flux tube.

There are also vortices in Laughlin's ground state wavefunction. The only difference is that instead of sitting at some point  $z_0$ , they are anchored to the particles themselves. To see this, consider  $\nu = \frac{1}{3}$ , and focus on the part involving just  $z_1$ :

$$\prod_{j>1} (z_j - z_1)^3 \quad (23)$$

If we freeze  $z_1$  and view it as a parameter like  $z_0$ , we see that there is a triple vortex on particle 1, and by symmetry, each particle. Apart from the vortex mandated by the Pauli principle, there are two vortices bound to each electron. (The Pauli zero comes with the turf, for being a fermion, and is not included in the count of vortices *attached* to enforce correlations.) The term vortex is again appropriate here since the location of the vortex is independent of any coordinate except, of course, the electron to which it is attached.

Contrast the zeros that constitute a vortex to the zeros of a generic antisymmetric analytic polynomial of the same degree as the Laughlin wavefunction. Once again, in  $\psi(z_1)$ , there is one zero at  $z_i$  ( $i > 1$ ) independent of all other

<sup>7</sup> The discussion is independent of this choice.

<sup>8</sup> The flux must be inserted slowly enough to prevent transitions to other excited states, but fast enough to prevent a lapse to the original ground state at the end, a possibility that exists in finite systems with disorder.



$z_j$  by the Pauli principle. This zero is part of a vortex, since all particles will "see" (in the wavefunctions  $\psi$ ) the Pauli zeros anchored on the other particles. The non-Pauli zeros of  $\psi(z_1)$ , by contrast, will be parametric functions of all other particle coordinates  $i = 2, \dots, N$  (what else can they be?). If we now consider  $\psi(z_2)$  its non-Pauli zeros will in general bear no relation to those of  $\psi(z_1)$ . Since the location of the non-Pauli zeros move parametrically with all the  $z$ 's, they do not belong to or form vortices. The reader must remember these distinctions in order to follow the discussion on the internal structure of CF's that follows shortly<sup>9</sup>.

Returning now to the quasihole, another important property is that it represents a charge deficit of  $1/(2s + 1)$  in electronic units. One way to show this is to employ the plasma analogy (Laughlin 1983a,b,1987). Here is a more general way (Su and Schrieffer, 1981, Su, 1984) that only depends on the state being gapped, incompressible, and having a quantized Hall conductance. As the flux quantum  $\Phi_0$  is adiabatically inserted to create the quasihole, the charge driven out to infinity is given by integrating the radial current density  $j$  produced by the Hall response to the induced azimuthal  $E$  field

$$\begin{aligned} Q &= - \int j(r, t) 2\pi r dt = \sigma_{xy} \int E 2\pi r dt \\ &= -\sigma_{xy} \int \frac{d\Phi}{dt} dt = -\Phi_0 \sigma_{xy} \end{aligned} \quad (24)$$

$$= -\frac{2\pi\hbar}{e} \frac{e^2}{2\pi\hbar} \nu = -e\nu. \quad (25)$$

For non-Laughlin fractions, the charge driven out by inserting one flux quantum is that of  $p$  quasiparticles, each of which has a charge equal to  $1/(2ps + 1)$  (Su, 1984). *Note that the fractional  $\sigma_{xy}$  is the cause behind the fractional charge.* The fractional charge of the quasiparticles has been confirmed experimentally (Goldman and Su, 1995, Goldman, 1996). The quasiparticles also have fractional statistics, as was pointed out by Halperin (1984), and explicitly confirmed in a wavefunction calculation by Arovas, Schrieffer, Wilczek, and Zee (1985)<sup>10</sup>.

The insensitivity of the Hall conductance to disorder in the FQHE can be established by an extension of the Laughlin-Halperin gauge arguments for the IQHE. To show this in toroidal geometry, the old arguments have to be supplemented by the fact that at fractions of the form  $p/q$ , there are  $q$  ground states, and the assumption that upon adding  $q$  flux quanta one returns to the starting state. There seems to be a consensus that Hall conductance is a ground state property that is impervious to a reasonable amount of disorder, very much the way superfluidity is.

Laughlin (1983a,b,1987) explained the plateaus as we move off the magic fraction as follows. Suppose we move slightly off  $\frac{1}{3}$ , to say  $\nu = \frac{1}{3} \pm .01$ . The system has two choices. It can either go to a FQHE ground state at this fraction, or it can be at  $\frac{1}{3}$  plus some number of quasiparticles (holes) of charge  $\pm\frac{1}{3}$ . It turns out the latter has a lower energy. The reason  $\sigma_{xy}$  is locked at  $\frac{1}{3}$  is that the quasiparticles/quasiholes, being sharply defined entities, get localized by the disorder potential and do not contribute to the Hall current. Thus the Hall conductance is expected<sup>11</sup> to be unaffected by disorder. As we move further off  $\frac{1}{3}$ , the system switches to another ground state with its own quasiparticles, and  $\sigma_{xy}$  jumps to the neighboring plateau. It is clear that if the fate of the quasiparticles/quasiholes is to get localized, their fractional charge cannot explain the fractional Hall conductance. As seen earlier, the connection is the other way around.

Thus a fairly complete description of the FQHE emerged from the Laughlin wavefunctions for the ground and quasiparticle/quasihole states. However numerous other issues surfaced and were tackled by subsequent work in the intervening two decades. We now steer our attention to those bearing directly on this paper.

Sticking for a moment to the Laughlin fractions, one important question that was raised was whether the system was really gapped, i.e., could there be lighter excitations than Laughlin's quasiparticles and quasiholes? Perhaps a quasiparticle and quasihole could bind to form a gapless excitation. Haldane and Rezayi (1985) verified by exact diagonalization of small systems that the  $\nu = \frac{1}{3}$  system was gapped. Girvin, MacDonald and Platzmann (1986) explored the question of neutral particle-hole excitations using an analogy with Feynman's work on superfluids and found a way to calculate the dispersion relations of the magnetoexciton within the LLL and showed that they were gapped in this sector for  $\nu = \frac{1}{3}$ .

<sup>9</sup> If the Laughlin function is perturbed slightly,  $\psi(z_1)$  will have one Pauli zero on every other electron and two others nearby. As long as the perturbation is small we can say, at length scales bigger than the excursion of the zeros, that electrons are bound to double vortices. For stronger perturbations that take us far from the Laughlin ansatz, there will be no simple description of correlations in terms of vortices.

<sup>10</sup> Fractional statistics was shown to be an emergent property of Composite Fermions by Goldhaber and Jain (1995).

<sup>11</sup> There is currently no way of rigorously calculating transport coefficients in the FQHE in the presence of disorder, hence the qualifier "expected".

## 2. Jain's Composite Fermions

Let us now proceed to the non-Laughlin fractions. Here the central question is this: Given that non-Laughlin fractions like  $\frac{2}{5}$  are seen in experiment, what is their wavefunction? Simply replacing the factors  $(z_i - z_j)^{2s+1}$  by  $(z_i - z_j)^{5/2}$  in Laughlin's ansatz is not acceptable since this does not produce the mandatory change of sign under particle exchange. In the hierarchy approach (Haldane, 1983, Halperin, 1983, 1984) the quasiparticles of the Laughlin states condense into their own FQHE states, whose quasiparticles in turn do the same thing and so on. While this approach gives a natural way to generate additional fractions, it does not give explicit wave functions in terms of electrons.

Explicit trial states for fractions  $\nu = \frac{p}{2ps+1}$  were provided by Jain (1989,1990,1994), who also explained why they were natural in terms of objects called *Composite Fermions* (CF's).

Jain's scheme hinges upon the seminal idea of flux attachment, which plays such a central role in the FQHE that it merits a little digression. It was introduced first by Leinaas and Myrheim (1977) in terms of wavefunctions, and explored in the language of Chern-Simons field theories by Wilczek (1982a,b). The crux of the idea is that in  $d = 2$  one can have particles (dubbed anyons by Wilczek) that suffer a phase change  $e^{i\theta}$  upon exchange, with  $\theta = 0, \pi \pmod{2\pi}$  corresponding to bosons and fermions respectively. To obtain one of these particles one takes a fermion and drives through its center a point flux tube, the amount of flux being decided by the desired  $\theta$ . In particular, if this tube contains an even/odd number of flux quanta, the composite particle one gets is a fermion/boson.

Jain (1989,1990) exploited flux attachment for  $\nu^{-1} = 2s + \frac{1}{p}$ , as follows. Suppose we trade our electrons for CF's carrying  $2s$  point flux quanta pointing opposite to the external  $B$ . On average the CF's effectively see  $\frac{1}{p}$  flux quanta per particle and fill up exactly  $p$  LL's. At the mean field level, this approach gives the following trial wavefunction:

$$\Psi_{p/2ps+1} = \prod_{i<j} \left[ \frac{(z_i - z_j)}{|z_i - z_j|} \right]^{2s} \cdot \chi_p(z, \bar{z}) \quad (26)$$

where  $\chi_p$  is the CF wavefunction with  $p$ -filled LL's and the prefactor takes the CF wavefunction back to the electronic wavefunction. (This will be made clearer shortly.)

Jain improved this ansatz in two ways and proposed:

$$\psi_{p/2ps+1} = \mathcal{P} \prod_{i<j} (z_i - z_j)^{2s} \cdot \chi_p(z, \bar{z}) \quad (27)$$

where

$$\prod_{i<j} (z_i - z_j)^{2s} \quad (28)$$

is called the *Jastrow factor*, and

$$\mathcal{P} : \bar{z} \rightarrow 2l^2 \frac{\partial}{\partial z} \quad (29)$$

projects the wavefunction to the LLL by eliminating all  $\bar{z}$ 's in  $\chi_p$  by turning them to  $z$ -derivatives on the rest of the wavefunction, except for the gaussian. (We shall see<sup>12</sup> that in the LLL  $[z, \bar{z}] = -2l^2$ .)

In making the change  $\prod_{i<j} \left[ \frac{(z_i - z_j)}{|z_i - z_j|} \right]^{2s} \rightarrow \prod_{i<j} (z_i - z_j)^{2s}$ , Jain replaces *flux tubes* by *vortices*. Although in both cases each particle picks up an extra phase shift of  $4\pi s$  on going around another, only the vortex factor has the desirable multiple zero that keeps the particles apart and lies in the LLL. Jain does not need to justify these modifications since in writing a trial wavefunction, he is free to resort to any changes that improve its energy. However, in the hamiltonian approach, the replacement of flux tubes by vortices and the projection to the LLL prove difficult but not insurmountable.

At  $p = 1$ , since  $\chi(z, \bar{z}) = \prod_{i<j} (z_i - z_j) \cdot \text{Gaussian}$ , we do not need  $\mathcal{P}$  to get back Laughlin's answer. For  $p > 1$  we have concrete expressions for  $\Psi$  in terms of electron coordinates, although the action of  $\mathcal{P}$  can be quite involved, and has a big impact on the wavefunction as we will see soon.

Thus while the degeneracy of the noninteracting problem is present for any  $\nu < 1$ , at the Jain fractions one can beat it by thinking in terms of Composite Fermions. As we move off the Jain fractions, the incremental CF's (particles

<sup>12</sup> For a nice review of states, operators, and matrix elements within the LLL see Girvin and Jach (1984)

or holes) get localized, giving rise to the plateaus. This is the sense in which the CF approach allows one to think of the FQHE in terms of the IQHE of the CF's. Since the IQHE can be understood without invoking interactions, it is sometimes suggested that CF's are free. Later we will see why this cannot be so.

The reader will have noted that both Laughlin and Jain wavefunctions *make no reference to the inter-electron potential*. This feature, which permits them to work for a whole class of potentials also renders them insensitive to specific features, including even the range of the interaction.

### 3. Vortices, zeros, and the dipole picture re-examined

Let us return to the vortices that are attached to electrons in the Jastrow factor. We have already discussed Halperin's observation (Halperin 1983, 1984) that electrons are bound to vortices in the Laughlin wavefunction. This concept was taken up and vigorously pursued by Read (1989). In particular Read made a clear distinction between flux attachment and vortex attachment, which were often loosely interchanged. Attaching electrons to flux tubes is a mathematical trick; flux tubes are unphysical and neutral. Vortices on the other hand are physical excitations of the Laughlin ground states, and electrons would naturally bind to them since the two are oppositely charged. The reason the CF sees a weaker field is not due to any mysterious capture of flux tubes. Indeed, the external field is uniform and not quantized into point flux tubes. What really happens in the Laughlin case is that each electron pairs with  $2s$  vortices and when the composite object goes around on a loop, it sees a phase change of  $-2\pi(2s+1)$  per enclosed particle due to the external  $B$  and  $2\pi(2s)$  per enclosed CF due to the vortices attached to each one.

Next, since the  $2s$ -fold vortex has a charge  $-2s/(2s+1)$  as per the flux insertion argument (Su, 1984), Eqn. (25), the vortices reduce  $e$  down to the quasiparticle or CF charge

$$e^* = 1 - \frac{2s}{2s+1} = \frac{1}{2s+1}. \quad (30)$$

In a Fermi liquid an added electron is also screened by a correlation hole; here the difference is that the possible response of the Laughlin liquid to an extra electron is limited to an integer number of vortices of quantized charge.

Read (1994, 1996) next applied the notion of electrons binding to vortices to the case  $\nu = \frac{1}{2}$  and derived what is called the *dipole picture*. We now turn to a critical review of this concept. Since the dipole picture comes from the projection involved in the  $\nu = \frac{1}{2}$  wavefunction, it is instructive to start with understanding the impact of  $\mathcal{P}$  in a general Jain fraction where higher CF-LL's ( $p > 1$ ) are filled.

Consider  $N$  electrons at  $\nu = \frac{2}{5}$ . Prior to projection, each electron in the Jain wavefunction has a double vortex sitting on it due to the Jastrow factor. As we explained earlier, the term vortex is merited since every other particle sees it at the same place. At this stage it is correct to view the CF as an electron plus a double vortex<sup>13</sup>. All this changes after projection to the LLL. Now there can only be  $5N/2$  zeros of the wavefunction as a function of, say,  $z_1$ ;  $N$  of them must lie on other electrons by the Pauli principle (forming single vortices anchored to electrons), with  $3N/2$  left over. Clearly they cannot all be on electrons, or be associated with them uniquely (since we have  $3/2$  zeros per electron). If the zeros are not on electrons, their location must depend parametrically on the locations of *all* the electrons and they cannot organize themselves into vortices.

Thus the wavefunction analysis leaves us with the following quandry. The CF cannot be viewed as an electron-vortex complex in the projected non-Laughlin states, but the quasiparticle charge,  $e^* = 1 - \frac{2ps}{2ps+1}$  is robust under  $\mathcal{P}$  since  $e^*$  is tied to  $\sigma_{xy}$  which is presumed robust under projection. *What, if any, is the entity that binds to the electron to bring  $e$  down to  $e^*$ ?* How does this entity enter the theory? What makes it bind to the electron? We will see that the EHT answers such questions.

Let us now return to  $\nu = \frac{1}{2}$ . In the unprojected wavefunction the double-vortex charge fully cancels the electron charge so that  $e^* = 0$ . Read (1994) has argued that the neutral CF of momentum  $k$  has a dipole moment  $d^* = kl^2$ , based the wave function at  $\nu = \frac{1}{2}$ , also called the Rezayi-Read (Rezayi and Read, 1994) wavefunction, in which  $\chi_p$  in Eqn. (27) is replaced by the filled Fermi sea:

$$\Psi_{\frac{1}{2}} = \mathcal{P} \prod_{i < j} (z_i - z_j)^2 \text{Det} |e^{i\mathbf{k}_i \cdot \mathbf{r}_j}|. \quad (31)$$

He considers

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<sup>13</sup> There is also another zero, generally non-analytic, in  $\chi_p$  which is antisymmetric. This is not part of the vortex count.

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \exp \frac{i}{2}(k\bar{z} + \bar{k}z) \quad k = k_x + ik_y. \quad (32)$$

Since under the action of  $\mathcal{P}$ ,  $\bar{z}$  acts on the analytic part of the wavefunction as  $2l^2\partial/\partial z$ ,  $e^{il^2k\frac{\partial}{\partial z}}$  causes the shift  $z \rightarrow z + ikl^2$  in the Jastrow factor. This motion of the vortex off the electron produces the dipole moment  $d^* = kl^2$ . The energy needed to separate the vortex from the electron (the Coulomb attraction), must begin with a term quadratic in the separation or momentum. This then gives the LLL fermions a kinetic energy or an effective mass  $m^*$  resulting from interactions.

Attractive though this picture is, closer scrutiny reveals that the preceding line of reasoning is incorrect in the following ways:

- Since *every*  $z_i$  gets translated,  $(z_i - z_j)^{2s} \rightarrow (z_i - z_j + ik_i - ik_j)^{2s}$  and particle  $i$  sees a multiple zero associated with  $z_j$  at  $z_j + i(k_j - k_i)$ . We cannot call this a vortex since the location of the zero varies with the label  $i$ . Different particles see it in different places. This problem is actually moot because of the next, more serious one.
- *Even this multiple zero is there for one particular assignment of  $k$ 's, or one term in the determinant. Upon antisymmetrization, we cannot relate the zeros of the sum over the  $N!$  terms to the zeros of individual terms.* The situation would have been different if we had been talking about poles which can survive such a sum. In fact, one of the two zeros that moved off the particles (in each individual term) should return upon antisymmetrization to lie exactly on the particles by Pauli's principle, leaving another zero to depend parametrically on all other coordinates, and all the  $k_i$ . Thus there is no reason to expect any simple relation between the location of the electrons and the non-Pauli zeros in  $\Psi_{\frac{1}{2}}$  or to conclude that these zeros form vortices. We commend to the reader experimentation with the limited but nonetheless instructive case of just two particles, to see some of these ideas in concrete form.
- If one looks at  $\nu = \frac{1}{2}$  in isolation, the fact there is one non-Pauli zero per electron (in the thermodynamic limit) may tempt one to suggest that perhaps in this case they organize themselves into vortices. Besides our analysis of the gapped fractions which shows that this is extremely unlikely, there is the general argument that zeros that do not lie on particles or external flux tubes (i.e., locations with an independent reality) must vary parametrically with all coordinates and therefore cannot be organized into vortices.

Thus at  $\nu = \frac{1}{2}$ , there seem to be two choices: Either use the unprojected wavefunction in which the Jastrow factor explicitly has two vortices per electron, but the dipole moment is zero (since the vortices are on the electrons), or go the LLL where the vortices disintegrate into a smaller number of ordinary zeros, not correlated with the electrons in any simple fashion. So what happens to the dipole?

Our EHT, which provides an operator realization of the CF, will show that the dipole picture is more robust than the wavefunction-based arguments pointing to it. However one must look for it not in the wavefunction, but in operators and correlation functions. Even then, the value of  $d^*$  will be sensitive to the details of the wavefunction. In this respect, it is quite unlike  $\sigma_{xy}$  which is robust under changes in the ground state wavefunctions, (including projection to the LLL), and depends only on the filling factor.

## E. Hamiltonian approach

Let us begin by reviewing the challenges facing any hamiltonian theory. In the FQHE there is more than enough room in the LLL for all the electrons, leading to a macroscopic number of degenerate ground states in the absence of interactions. This preempts many of the standard approximations. It is expected that when interactions are turned on, there will emerge from this degenerate manifold a unique ground state, separated by a gap from other low-lying excited states. While the true ground state and low-lying excitations will of course contain an admixture of higher LL's, one expects that in the limit of bare mass  $m \rightarrow 0$ , or  $\omega_0 \rightarrow \infty$ , there will emerge a low-energy sector spanned by LLL states. *In other words, in the limit  $m \rightarrow 0$  a nonsingular low-energy theory must emerge.* The  $m$ -independent Laughlin wavefunctions for the ground state, quasiparticles and quasiholes illustrate this point.

If all memory of  $m$  is lost, the only energy scale is set by the interactions. How is one to isolate the LLL physics, staring with the full Hilbert space? How is one to battle the degeneracy of the noninteracting ground state? How is one to get rid of the  $1/m$  dependence of the kinetic energy? These are problems for the hamiltonian approach.

Next, there is one vestige of  $m$ -dependence even in the LLL, discovered by Simon, Stern and Halperin (1996): if the external field  $B$  varies slowly in space, the zero point energy  $eB/2m$ , can no longer be eliminated by redefinition

of the zero. In fact the particles behave as if they have a magnetic moment  $\mu^* = e/2m$  coupled to  $B$ . The theory must reproduce this moment.<sup>14</sup>

We also do not want to ban higher LL's too soon, since the Hall conductance necessarily involves them<sup>15</sup>. Finally, Kohn's theorem (Kohn, 1961) assures us that despite interactions, the cyclotron mode at  $q = 0$  must be at  $\omega_0 = eB/m$  and it must saturate certain sum rules at small  $q$ . The theory must pass this test.

*The hamiltonian theory must succeed in keeping the  $m$ -dependence where it belongs and exorcise it elsewhere.*

The next set of issues concern the quasiparticles, the CF's. A heuristic picture that we saw arising from the unprojected wavefunctions is that the quasiparticles are composites of electrons and  $2s$  vortices. How is this actually realized in the hamiltonian approach? After all, this is not analogous to the statement that mesons are made of quarks and antiquarks, for unlike antiquarks, which appear in the hamiltonian, vortices do not. Vortices are not independent of electrons and to speak of them and electrons at the same time surely paves the way for overcounting. In the non-Laughlin cases, the issue is further complicated by the fact that there are not enough zeros to form  $2s$  vortices per electron, and in any case the zeros are not organized into vortices. Yet some object of the same charge as the  $2s$ -fold vortex seems to bind with the electron since in the end  $e$  gets reduced to  $e^*$ . Somehow this object has to enter the theory and give rise to the CF.

Other issues surround the CF. How strong are interactions between CF's? Why do they sometimes appear to be free when we can give persuasive arguments for why they cannot be? How is disorder to be included?

Besides addressing these questions of principle, we need answers for *quantitative questions*. For example, what is the gap at  $\nu = \frac{1}{3}$ ? Now, very precise answers can be given for such a question in the trial wavefunction approach<sup>16</sup> or by exact diagonalization<sup>17</sup>. While these results are founded on the microscopic hamiltonian, the intermediate steps are computer-intensive. It would be nice to have a theory which displays transparently the features of the CF deduced from the study of wave functions, and furnishes quantitative results, to say 10% accuracy. The same goes for polarizations in various states, and transitions between them.

A specially interesting set of questions appear at and near  $\nu = 1/2$ . Can the notion of the CF survive in this region without a robust gap? In particular, will they really behave like particles of charge  $e^*$  in this region? Since the average effective magnetic field  $B^* = 0$  at  $\nu = 1/2$ , we have fermions in zero magnetic field, except for fluctuations. Will this Fermi system form a Fermi liquid after including fluctuations? What are its response functions? Will it be compressible? If so, how do particles with  $e^* = 0$  manage to be compressible or even have a nonzero Hall conductance? What are the collective modes and how can they be detected? How is disorder to be handled? Exact diagonalization and the wavefunction based approaches are not very helpful for unequal-time correlations, and finite-temperature physics. Hamiltonian theories should fill this void and answer all the questions raised above.

## F. Organization of this review

Section II introduces the reader to some notation and the physics of the LLL, including Kohn's theorem which is often referred to in our work. Section III describes the Chern-Simons theory of flux attachment including both composite bosons and fermions. Section IV deals with the region at and near  $\nu = 1/2$  and is mainly about the work of Halperin, Lee and Read (1993). Section V explains our extended hamiltonian approach, EHT, formulated in an enlarged Hilbert space. It is shown that there are two approaches to solving our final equations: the conserving approximation in which the constraints are respected, but the CF physics is hidden, and a shortcut in which the opposite is true. The compressibility paradox at  $\nu = 1/2$  is discussed. Section VI illustrates the conserving approximation in the EHT via the calculation of the structure factor at small  $q$  in the Jain series and the magnetoexciton dispersion relations for  $1/3$  and  $2/3$ . Section VII is devoted to the computation of gaps and comparison to numerical work and experiment. Sections VIII deals with magnetic phenomena at  $T = 0$  for gapped and gapless systems. The question of whether CF's are free or not is discussed and answered in the negative, along with an explanation of why sometimes they appear to be so. Section IX addresses physics at  $T > 0$ , describing the computation of polarization and relaxation at  $\nu = \frac{1}{2}$  and polarization at  $\frac{1}{3}$  and  $\frac{2}{5}$ . The results are then compared to experiment. Section X deals with inhomogeneous states. Section XI gives a critical evaluation of the EHT. A summary and a discussion of open problems follow in Section XII.

<sup>14</sup> This magnetic moment is actually a compact way to describe an effect that is orbital in origin.

<sup>15</sup> The current operator goes as  $1/m$  while  $\sigma_{xy}$  does not. The  $1/m$  in the current is cancelled by the energy denominator of order  $\omega_0$  that comes from virtual transitions to the higher LL's. For a more detailed explanation, see Girvin, MacDonald, and Platzman (1986), and Sondhi and Kivelson (1992).

<sup>16</sup> For a review of this method with complete references, see Jain and Kamilla (1998).

<sup>17</sup> Due to limitations of space we present only the earliest references: Girvin and Jach (1983), Haldane and Rezayi (1985), Su (1985), Haldane (1985), Yoshioka (1986), Morf and Halperin (1986, 1987), with the exception of Morf *et al* (2002).

## II. PRELIMINARIES AND NOTATION

This section focuses on a single electron in two dimensions. The hamiltonian is

$$H_0 = \frac{(\mathbf{p} + e\mathbf{A})^2}{2m} = \frac{\Pi^2}{2m}. \quad (33)$$

Here  $\mathbf{A}$  is the vector potential that leads to the external magnetic field  $\mathbf{B} = \nabla \times \mathbf{A} = -B\hat{\mathbf{z}}$ ,  $\mathbf{p}$  is the canonical momentum of the electron,  $-e$  is its charge, and  $m$  is its bare or band mass. Note that  $\mathbf{B}$  points along the negative  $z$ -axis.

Although one needs to pick a gauge for  $\mathbf{A}$  in order to find the wavefunctions, we can obtain the spectrum without making that choice. Let us define a *cyclotron coordinate*

$$\boldsymbol{\eta} = l^2 \dot{\mathbf{z}} \times \boldsymbol{\Pi} \quad (34)$$

where  $l = \sqrt{\hbar/eB}$  is the magnetic length. Despite the name, the two components of  $\boldsymbol{\eta}$  are not commuting but canonically conjugate variables:

$$[\eta_x, \eta_y] = i l^2. \quad (35)$$

It follows that

$$H_0 = \frac{\boldsymbol{\eta}^2}{2ml^4} \quad (36)$$

describes a harmonic oscillator with energies

$$E = (n + \frac{1}{2})\omega_0 \quad (37)$$

where  $n$  is the LL index.

The LL's are highly degenerate because another conjugate pair that commutes with  $\boldsymbol{\eta}$ , and called the *guiding center coordinate*

$$\mathbf{R} = \mathbf{r} - \boldsymbol{\eta} \quad (38)$$

is cyclic. The components of  $\mathbf{R}$  obey

$$[R_x, R_y] = -i l^2. \quad (39)$$

Thus  $l^2$  plays the role of  $\hbar$ . Since in the LLL  $\langle \eta^2 \rangle = l^2$ ,  $\mathbf{R}$  roams all over the sample, whose area  $L^2$  plays the role of phase space. The degeneracy of the LLL (or any LL) is, from the Bohr-Sommerfeld quantization rule,

$$D = \frac{L^2}{\hbar} = \frac{L^2}{2\pi l^2} = \frac{eBL^2}{2\pi\hbar} = \frac{\Phi}{\Phi_0} \quad (40)$$

where  $\Phi_0$  is the flux quantum. This leads to the following result worthy of committing to memory:

$$\frac{\text{LLL states}}{\text{particles}} = \frac{D}{N} = \frac{\text{flux quanta density}}{\text{particle density}} = \frac{eB}{2\pi n\hbar} \quad (41)$$

At points where steps in  $\sigma_{xy}$  cross the straight line dictated by Galilean invariance,

$$\sigma_{xy} = \frac{ne}{B} = \frac{e^2}{2\pi\hbar} \nu = \frac{e^2}{2\pi\hbar} \frac{p}{2ps + 1}, \quad (42)$$

and

$$\frac{\text{LLL states}}{\text{particles}} = \frac{\text{flux quanta density}}{\text{particle density}} = \frac{eB}{2\pi\hbar n} = 2s + \frac{1}{p} = \nu^{-1}. \quad (43)$$

Hereafter we set  $\hbar = 1$ .

### A. Gauge choices

There are two famous choices for gauge. In the *Landau gauge*  $\mathbf{A} = -\mathbf{j}Bx$  the Hamiltonian is cyclic in  $y$ , and hence has the eigenfunctions  $e^{iky}$ . Making the ansatz

$$\psi(x, y) = e^{iky}\phi(x) \quad (44)$$

we find that  $\phi$  obeys the equation for a displaced harmonic oscillator

$$-\frac{1}{2m}\frac{d^2\phi}{dx^2} + \frac{1}{2}m\omega_0^2(x - kl^2)^2\phi = E\phi \quad (45)$$

The LLL corresponds to putting the oscillator in its ground state. The degeneracy of any LL can be computed by considering a sample with sides  $L_x$  and  $L_y$ , with periodic boundary conditions in the  $y$  direction. This forces  $k = \frac{2\pi j}{L_y}$ . Since the wavefunction is centered around  $x = kl^2 < L_x$ , we must demand  $0 \leq j \leq \frac{L_x L_y}{2\pi l^2}$ . Thus the degeneracy  $D$  of each LL, is (going back to a square sample)  $D = \frac{BL^2}{\Phi_0}$ , in agreement with our prior gauge-invariant counting of states. We shall use this gauge in the computation of the magnetoexciton.

In the rest of this article we employ the *symmetric gauge* in which

$$\mathbf{A} = \frac{eB}{2}(\mathbf{i}y - \mathbf{j}x) \quad (46)$$

$$\boldsymbol{\eta} = \frac{1}{2}\mathbf{r} + l^2\hat{\mathbf{z}} \times \mathbf{p} \quad (47)$$

$$\mathbf{R} = \frac{1}{2}\mathbf{r} - l^2\hat{\mathbf{z}} \times \mathbf{p}. \quad (48)$$

### B. Getting to know the LLL

As with any simple harmonic oscillator, we can construct ladder operators from the canonically conjugate pair  $\eta_x$  and  $\eta_y$  (with  $l^2$  playing the role of  $\hbar$ )

$$a_\eta = \frac{1}{l\sqrt{2}}(\eta_x + i\eta_y) \quad (49)$$

$$a_\eta^\dagger = \frac{1}{l\sqrt{2}}(\eta_x - i\eta_y) \quad (50)$$

The LLL condition  $a_\eta|LLL\rangle = 0$  gives

$$\psi = e^{-|z|^2/4l^2}f(z) \quad (51)$$

where  $z = x + iy$ . A basis for  $\psi$  is

$$\psi_m(z) = z^m e^{-|z|^2/4l^2} \quad m = 0, 1, \dots \quad (52)$$

The Gaussian is often suppressed. The state has angular momentum  $L_z = m$ .

If  $\nu = 1$ , (one electron per LLL state) there is a unique noninteracting ground state, which may then be perturbed by standard means,

$$\chi_1 = \prod_{i < j} (z_i - z_j) \cdot \text{Gaussian} = \text{Det} \begin{vmatrix} z_1^0 & z_1^1 & z_1^2 & \dots \\ z_2^0 & z_2^1 & z_2^2 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \cdot \text{Gaussian} \quad (53)$$

For  $\nu < 1$ , we often want to focus on the limit  $\omega_0 \rightarrow \infty$  and work entirely within the LLL. If in  $H = T + V$  we set  $T$  equal to a constant ( $eB/2m$  per particle), all the action is in  $V$ . Why is this a problem if  $V$  is a function of just coordinates? After all,

$$\rho(\mathbf{r}) = \sum_j \delta(\mathbf{r} - \mathbf{r}_j) \quad (54)$$

and

$$V = \frac{1}{2} \int d^2r \int d^2r' \rho(\mathbf{r}) v(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') \quad (55)$$

$$= \frac{1}{2} \sum_{\mathbf{q}} \rho(\mathbf{q}) v(\mathbf{q}) \rho(-\mathbf{q}) \quad (56)$$

$$\rho(\mathbf{q}) = \int d^2r \rho(\mathbf{r}) e^{-i\mathbf{q} \cdot \mathbf{r}} = \sum_j e^{-i\mathbf{q} \cdot \mathbf{r}_j}. \quad (57)$$

The point is that if one sets  $T = eB/2m$ , the LLL value, one must project the operator  $\mathbf{r}$  to the LLL. The coordinates  $x$  and  $y$ , which commute in the full Hilbert space, no longer commute in the LLL and  $V$  now contains noncommuting operators.

### C. Projection to the LLL

Let  $\mathcal{P}$  denote projection to the LLL<sup>18</sup>. Then

$$\mathcal{P} : \mathbf{r} = \mathbf{R} + \boldsymbol{\eta} \Rightarrow \mathbf{R}. \quad (58)$$

The projected components do not commute:

$$[R_x, R_y] = -il^2 \quad \text{or} \quad [z, \bar{z}] = -2l^2 \quad \text{in the LLL.} \quad (59)$$

As for the densities

$$\mathcal{P} : e^{-i\mathbf{q}\cdot\mathbf{r}} \Rightarrow \langle e^{-i\mathbf{q}\cdot\boldsymbol{\eta}} \rangle_{LLL} e^{-i\mathbf{q}\cdot\mathbf{R}} = e^{-q^2 l^2 / 4} e^{-i\mathbf{q}\cdot\mathbf{R}} \quad (60)$$

Thus the projected problem is defined by

$$\bar{H} = \frac{1}{2} \sum_{\mathbf{q}} e^{-q^2 l^2 / 2} \bar{\rho}(\mathbf{q}) v(q) \bar{\rho}(-\mathbf{q}) \quad (61)$$

$$\bar{\rho}(\mathbf{q}) = \sum_j e^{-i\mathbf{q}\cdot\mathbf{R}_j} \quad (62)$$

The projection  $\bar{\rho}$  is the *Magnetic Translation Operator*<sup>19</sup>, which differs from the projected density by a factor  $e^{-q^2 l^2 / 4}$ . We shall often refer to it as the density, but take care to include the gaussian in Eqn.(61). The commutation rules of  $\bar{\rho}$  define the magnetic translation algebra:

$$[\bar{\rho}(\mathbf{q}), \bar{\rho}(\mathbf{q}')] = 2i \sin \left[ \frac{(\mathbf{q} \times \mathbf{q}') l^2}{2} \right] \bar{\rho}(\mathbf{q} + \mathbf{q}') \quad (63)$$

which was thoroughly exploited in the work of Girvin, MacDonald, and Platzman (1986).

There is no small parameter in  $\bar{H}$  and the overall energy scale is set by  $v(q)$ . This is why the FQHE problem is unique. As mentioned earlier the Hartree-Fock solution is also not an option at fractional filling.

### D. Kohn's Theorem

By Kohn's theorem we shall mean the following results. Consider, in a translationally invariant system, the density-density response function  $K(\mathbf{q}, \omega)$ , the Fourier transform of

$$K(\mathbf{q}, t) = i\theta(t) \langle 0 | [\rho(\mathbf{q}, t), \rho(-\mathbf{q}, 0)] | 0 \rangle. \quad (64)$$

In the frequency domain we define

$$K(\mathbf{q}, \omega) = \int_{-\infty}^{\infty} K(\mathbf{q}, t) e^{i\omega t} dt. \quad (65)$$

From just the canonical commutation rules it can be shown that  $K$  obeys the sum rule

$$\int_0^{\infty} \text{Im} K(\mathbf{q}, \omega) \frac{d\omega}{\pi} = \frac{q^2 n}{2m}. \quad (66)$$

<sup>18</sup> A very nice introduction to LLL physics appears in Girvin and Jach (1984)

<sup>19</sup> These operators and the projective group they form have a long history, and were first used to describe symmetries of the noninteracting electron Hamiltonian in a magnetic field. For references, see: Peterson (1960), Brown (1964), Zak (1964a,b). To the best of our knowledge, Girvin, MacDonald, and Platzman (1986) were the first to concentrate on the *algebra* of this operator



Kohn showed that  $K$  must have a pole, the magnetoplasmon, at the cyclotron frequency  $\omega_0 = eB/m$  with a residue that *saturates* the above sum rule as  $q \rightarrow 0$ .

It follows that  $\bar{\rho}(\mathbf{q})$ , the restriction of  $\rho(\mathbf{q})$  to the LLL, cannot have (transition) matrix elements that are linear in  $q$  for small  $q$ .

The structure factor  $S(q, \omega)$

$$S(q, \omega) = \sum_n |\langle 0 | \rho(q, 0) | n \rangle|^2 \delta(\omega - E_n) \quad (67)$$

is related to  $K(q, \omega)$  for  $\omega > 0$  by

$$\frac{1}{\pi} \text{Im } K(q, \omega) = S(q, \omega). \quad (68)$$

<sup>20</sup> Kohn theorem tells us that if we limit ourselves to the LLL,  $S(q) \simeq q^4$  for small  $q$ .

### III. HAMILTONIAN THEORY I - THE CS APPROACH

The hamiltonian for electrons in a vector potential  $\mathbf{A}$  is

$$H = \sum_j \frac{(\mathbf{p}_j + e\mathbf{A}(\mathbf{r}_j))^2}{2m} + V \quad (69)$$

where  $V$  is the electron-electron interaction, say Coulombic. Disorder is not included. As stated earlier, the field  $\mathbf{A}$  is such that there are  $2s + \frac{1}{p}$  flux quanta or LLL states per electron and the attendant degeneracy frustrates perturbative analysis.

The first step in the CS approach is to deal with the degeneracy by resorting to flux attachment. For the Laughlin fractions there are actually *two* options involving either composite bosons (CB) or composite fermions.

#### A. Composite bosons

Historically, the first treatment came from Zhang, Hansson and Kivelson (1989) who considered Laughlin fractions  $\nu = 1/(2s + 1)$ . They traded electrons for CB's carrying  $2s + 1$  flux quanta in opposition to the applied field so that at mean-field level the bosons saw zero field and had a unique ground state<sup>21</sup>. The trading is done by introducing a Chern-Simons wave function  $\Psi_{CS}$  defined as follows:

$$\Psi_e = \prod_{i < j} \frac{(z_i - z_j)^{2s+1}}{|z_i - z_j|^{2s+1}} \Psi_{CS} \equiv \exp((2s + 1)i \sum_{i < j} \phi_{ij}) \Psi_{CS}. \quad (70)$$

$$H_{CS} = \sum_i \frac{(\mathbf{p}_i + e\mathbf{A}(\mathbf{r}_i) + \mathbf{a}_{cs}(\mathbf{r}_i))^2}{2m} + V \quad (71)$$

where  $\phi_{ij}$  is the phase of the coordinate difference  $z_i - z_j$ . Since  $\Psi_e$  changes sign under particle exchange and the prefactor produces  $2s + 1$  extra minus signs,  $\Psi_{CS}$  describes bosons.

The CS gauge field,  $\mathbf{a}_{cs}$ , comes from the action of  $\mathbf{p}$  on the prefactor (which is just the phase of the Jastrow factor) that multiplies  $\Psi_{CS}$ :

$$\mathbf{a}_{cs}(\mathbf{r}_i) = 2s \nabla \sum_{j \neq i} \phi_{ij} \quad (72)$$

$$\oint \mathbf{a}_{cs}(\mathbf{r}_i) \cdot d\mathbf{r}_i = 2s \oint \sum_{j \neq i} \nabla \phi_{ij} \cdot d\mathbf{r}_i \quad (73)$$

$$= 2\pi(2s + 1) \text{ (number of particles enclosed)} \quad (74)$$

$$\nabla \times \mathbf{a}_{cs} = 2\pi(2s + 1)\rho \quad (75)$$

<sup>20</sup> To see this, introduce a complete set of exact eigenstates  $|n\rangle$  of  $H$  between the two factors of  $\rho$  in Eqn. (64), express the Heisenberg operator  $\rho(t)$  in terms of  $\rho(0)$  and  $H$ , and do the time integral.

<sup>21</sup> This mean-field idea was first applied to anyon superconductivity by Laughlin (1988). Many more works on this topic followed: Fetter, Hanna, and Laughlin (1989), Hanna, Laughlin, Fetter (1989), Chen, Halperin, Wilczek, and Witten (1989), Halperin, March-Russell, and Wilczek (1989), Lee and Fisher (1989), Hanna, Laughlin, Fetter (1991), Dai, Levy, Fetter, Hanna, and Laughlin (1992).

Eq. (75) shows explicitly that the flux quanta density is  $2s + 1$  times the particle density. This is what one means by flux attachment. Eqs. (71) and (75) define a Chern-Simons theory.<sup>22</sup> The possibility that the FQHE would be described by a CS theory was presaged by Girvin (1987).

Since the idea of flux attachment is to cancel the applied field on average, Zhang, Hansson, and Kivelson separate  $\mathbf{a}_{cs}$  and  $\rho$  into average and fluctuating parts:

$$\nabla \times \langle \mathbf{a}_{cs} \rangle + \nabla \times : \mathbf{a}_{cs} := 2\pi(2s + 1)n + 2\pi(2s + 1) : \rho : \quad (76)$$

This gives

$$\begin{aligned} H_{CS} &= \sum_i \frac{(\mathbf{p} + e\mathbf{A} + \langle \mathbf{a}_{cs} \rangle + : \mathbf{a}_{cs} :)^2_i}{2m} + V \\ &= \sum_i \frac{(\mathbf{p} + : \mathbf{a}_{cs} :)^2_i}{2m} + V \end{aligned} \quad (77)$$

upon using the fact that the flux due to  $e\mathbf{A}$  precisely cancels that due to  $\langle \mathbf{a}_{cs} \rangle$ . Bosons in zero field have no degeneracy problem (assuming they have some repulsive interactions) and allow one to describe much of the FQHE physics in the familiar language of superfluids. For example, it has been shown by computing response functions, that the superfluidity of the bosons implies the FQHE for electrons (Zhang, Hansson, and Kivelson, 1989), that the vortex in the superfluid is Laughlin's quasi-hole (Lee and Zhang, 1991) The nature of the collective modes has also been explored (Kane, Kivelson, Lee, and Zhang, 1991).

Neglecting  $: \mathbf{a}_{cs} :$  (the mean-field approximation) and the interaction,  $\Psi_{CS}$  the wavefunction for bosons (in Eqn. (70)) is just unity and that of electrons is

$$\Psi_e = \prod_{i < j} \left( \frac{z_i - z_j}{|z_i - z_j|} \right)^{2s+1} \cdot 1 \quad (78)$$

which is the phase of Laughlin's answer. Kane, Kivelson, Lee, and Zhang (1991) showed that if long-wavelength gaussian fluctuations are included, the full Laughlin wavefunction is obtained at long distances. But the same fluctuations also reduce the long-range order in the boson field down to the power-law order found by Girvin and MacDonald (1987) who analyzed a gauge-transformed version of the Laughlin wavefunction.<sup>23</sup> Read (1989) then showed that for Laughlin fractions, one could form an operator (which was a composite of an electron and  $2s + 1$  vortices) which was neutral and had true long-range order. The corresponding Landau-Ginzburg theory for the order parameter was, however, very complicated. Constraints of time and space prevent us from describing CB's any further. We refer the reader to primary sources and excellent reviews (Zhang, 1992, Karlhede, Kivelson, and Sondhi, 1993).

There are, however, some shortcomings in the CB approach. First, it is restricted to Laughlin fractions. Next, since bosons have to be interacting to be stable, a noninteracting starting point does not exist. Finally, there is singular dependence on  $m$  as  $m \rightarrow 0$  and it is hard to carry out quantitative computations. Nonetheless, this work has served as a paradigm for the hamiltonian approach.

## B. Composite fermions

We turn now to composite fermions. It was seen in the review of Jain's work that by trading electrons for CF's which carry  $2s$  flux quanta in opposition to the applied field, one could get a fermionic system that on the average sees a field  $\mathbf{A}^*$  that is just right to fill  $p$  Landau levels. Lopez and Fradkin (1991,1992,1993,1998) were the first to accomplish this in the hamiltonian approach.<sup>24</sup> They traded the electronic wavefunction  $\Psi_e$  for  $\Psi_{CS}$  defined as follows:

$$\Psi_e = \prod_{i < j} \frac{(z_i - z_j)^{2s}}{|z_i - z_j|^{2s}} \Psi_{CS} \equiv \exp(2is \sum_{i < j} \phi_{ij}) \Psi_{CS}. \quad (79)$$

<sup>22</sup> Note that these manipulations could just as well be done in the path integral formulation. We have chosen to use the first quantized operator version for all our discussions, in the interest of uniformity.

<sup>23</sup> This is to be expected given that the gauge transformation of Girvin and MacDonald (1987) is the same CS transformation of Eq. (70).

<sup>24</sup> Note that we do not distinguish between the functional integral formalism which they employed, and the operator approach which we used in our work, and in the interest of uniformity, throughout this paper.

$$H_{CS} = \sum_i \frac{(\mathbf{p}_i + e\mathbf{A}(\mathbf{r}_i) + \mathbf{a}_{cs}(\mathbf{r}_i))^2}{2m} + V. \quad (80)$$

Since  $\Psi_e$  describes fermions, so does  $\Psi_{CS}$  since the phase factor is even under particle interchange. The CS gauge field  $\mathbf{a}_{cs}$  now obeys

$$\nabla \times \mathbf{a}_{cs} = 4\pi s \rho. \quad (81)$$

Separating  $\mathbf{a}_{cs}$  and  $\rho$  into average and fluctuating parts

$$\begin{aligned} H_{CS} &= \sum_i \frac{(\mathbf{p} + e\mathbf{A} + \langle \mathbf{a}_{cs} \rangle + : \mathbf{a}_{cs} :)_i^2}{2m} + V \\ &= \sum_i \frac{(\mathbf{\Pi} + : \mathbf{a}_{cs} :)_i^2}{2m} + V \end{aligned} \quad (82)$$

$$\mathbf{\Pi} = \mathbf{p} + e\mathbf{A} + \langle \mathbf{a}_{cs} \rangle \equiv \mathbf{p} + e\mathbf{A}^* \quad (83)$$

$$\nabla \times (e\mathbf{A} + \langle \mathbf{a}_{cs} \rangle) = -eB + 4\pi ns \quad (84)$$

$$= -\frac{eB}{2ps + 1} \equiv -eB^* \quad (\mathbf{A}^* = \frac{\mathbf{A}}{2ps + 1}) \quad (85)$$

$$l^* = \frac{1}{\sqrt{eB^*}} = l\sqrt{2ps + 1} \quad (86)$$

The following important results emerged from the work of Lopez and Fradkin (1991,1992,1993):

- If we ignore  $: \mathbf{a}_{cs} :$  and  $V$ , the composite fermions see  $\frac{1}{p}$  flux quanta each (since  $2s + \frac{1}{p} \rightarrow \frac{1}{p}$  under flux attachment) and have a unique ground state  $\chi_p$  of  $p$  filled LL's. Excitations are given by pushing fermions into higher CF-LL's.

There is however a problem: If we excite a fermion from level  $p$  to  $p+1$ , the energy cost (activation gap) of the particle-hole pair is  $\Delta = eB^*/m$  plus corrections due to neglected terms. This divergent dependence on  $m$  flies in the face of the nonsingular  $m \rightarrow 0$  limit we have argued must exist<sup>25</sup>. We want  $\Delta \simeq e^2/\varepsilon l$  in the Coulomb case.

- At the mean-field level, the CF wavefunction  $\chi_p$ , transformed back to electrons is

$$\Psi_e = \prod_{i < j} \left( \frac{z_i - z_j}{|z_i - z_j|} \right)^{2s} \chi_p(z, \bar{z}) \quad (87)$$

Lopez and Fradkin showed that fluctuations at one loop give the square of the wavefunction (at long distances) for Laughlin fractions. (The factors  $|z_i - z_j|^{2s}$  in the denominator of the mean-field wave function are eliminated by fluctuations.)

- They calculated time-dependent density-density response functions in the Random Phase Approximation (RPA) (which will be explained in the next section). The cyclotron mode appears with the right position and residue. However, between the cyclotron mode ( $eB/m$ ) and the LLL excitations, there are many spurious modes attributable to the ubiquitous presence of  $m$  which prevents a clear separation of LLL and non-LLL energy scales. This is a problem common to all CS theories and was well appreciated by the authors.

The Lopez-Fradkin work paved the way for subsequent work to which we now turn.

#### IV. PHYSICS AT AND NEAR $\nu = \frac{1}{2}$

In the early days the CF, there was a widespread belief that its utility was confined to fractions with a robust gap: the all-forgiving gap allowed one to neglect, in a first approximation, interactions, disorder and gauge field fluctuations. It was therefore quite a surprise to see that the CF survived even when  $p \rightarrow \infty$  or  $\nu \rightarrow 1/2s$ , and the CF cyclotron gap  $eB^*/m = eB/(2ps + 1)m$  approached zero.

<sup>25</sup> Jain does not have this problem since he does not use  $H_{CS}$  or  $\chi_p$  or its excitations directly. For him the CS picture is a step towards getting electronic wavefunctions for the ground and excited states by attaching the Jastrow factor and projecting. The energy gap is computed as the difference in  $\langle V \rangle$  between the ground and excited electronic wavefunctions.

### A. Physics at $\nu = \frac{1}{2}$

Kalmeyer and Zhang (1992) were the first to discuss the case  $\nu = 1/2$ . Their work emphasized the following important point. One may expect that the effect of disorder will be rather small since the electron donors lie at a respectable distance from the electron gas itself. However any small charge inhomogeneity induced by disorder is accompanied by a corresponding flux density in a CS theory, and this can cause significant scattering.

A landmark study of the region at and near  $\nu = 1/2$  was made by Halperin, Lee and Read (1993), (HLR) who seriously pursued the remarkable possibility that a Fermi liquid could be hiding deep in the FQH regime. HLR started with the following hamiltonian at  $\nu = 1/2$ :

$$H_{CS} = \sum_i \frac{(\mathbf{p} + : \mathbf{a}_{CS} :)^2_i}{2m} + V \quad (88)$$

where  $: \mathbf{a}_{CS} :$  is related to charge *fluctuations* by the CS condition

$$\nabla \times : \mathbf{a}_{CS} := 4\pi : \rho : . \quad (89)$$

Note that CS theory dictates that  $m$  must be the bare mass, since flux attachment by *minimal* coupling is a gauge transformation performed on the bare electron. If one wants to take the view that this is an effective low-energy theory with an effective mass  $m^*$ , one must also be prepared for the possibility that the coupling of the fermion to the gauge field is more complicated.

The cornerstone of HLR's work is the computation of the electromagnetic response functions. Let us recall some general results on this topic so we may better understand and appreciate their work. In writing this description we were greatly aided by the reviews of Halperin (in Das Sarma and Pinczuk, 1997) and Simon (in Heinonen, 1998).

If an external four-potential  $eA_\mu^{ext}(\mathbf{q}, \omega)$  is applied to the system, it will generate a four-current  $j_\mu(\mathbf{q}, \omega)$  (whose components are the *number* current and density) as per

$$j_\mu(\mathbf{q}, \omega) = eK_{\mu\nu}(\mathbf{q}, \omega)A_\nu^{ext}(\mathbf{q}, \omega). \quad (90)$$

Linear response theory tells us that

$$K_{\mu\nu} = \int_{-\infty}^{\infty} dt e^{i\omega t} i\theta(t) \langle 0 | [j_\mu(\mathbf{q}, t), j_\nu(-\mathbf{q}, 0)] | 0 \rangle \quad (91)$$

where  $|0\rangle$  is the vacuum state.

For pedagogical purposes consider just  $K_{00}$ , the density-density correlator. In free-field theory,  $K_{00}$  is given by  $K_{00}^0$ , the particle-hole bubble, in which the particle and hole created by one  $\rho$  are absorbed by the other. The full theory of course requires us to include interactions. Once again, for pedagogical purposes, let us begin with the case with just the Coulomb interactions. The exact  $K_{00}$  is given by an infinite sum of Feynman graphs in which the electrostatic propagator  $v(q) = 2\pi e^2/q$  appears in all possible ways. It is possible to organize the sum as in Figure (3). Each bubble  $K_{00}^v$ , called the *Coulomb-irreducible response*, has the property that it cannot be cut into two disjoint pieces by snipping just one Coulomb propagator  $v(q)$ , denoted by the wiggly line connecting the irreducible bubbles. Performing the geometric sum one finds

$$K_{00} = \frac{K_{00}^v}{1 + v(q)K_{00}^v} = \frac{1}{[K_{00}^v]^{-1} + v(q)}. \quad (92)$$

The response  $K_{00}^v$  has the following significance. Consider some conducting system with an applied potential  $e\phi^{ext}$ . An electron inside the conductor feels in addition the potential generated by the charges themselves, i.e., it feels the total potential  $e\phi^T = e\phi^{ext} - v(q)\rho(\omega, \mathbf{q})$ , and  $K_{00}^v$  is the response to this total field. To verify this let us write

$$\rho(\omega, \mathbf{q}) = eK_{00}^v [e\phi^{ext} - v(q)\rho(\omega, \mathbf{q})] \quad (93)$$

solving which we get

$$\rho(\omega, \mathbf{q}) = \frac{eK_{00}^v}{1 + v(q)K_{00}^v} \phi^{ext}. \quad (94)$$

Who cares about the total field? Answer: The voltmeter we use to measure the drop in a wire. The voltmeter responds to the total electric field and not to the externally applied one that would exist in the absence of the conductor. The longitudinal conductivity  $\sigma_{xx}$  can be related to  $K_{00}^v$  as per

$$\sigma_{xx} = \frac{ej}{ET} = e^2 \frac{\omega}{iq^2} K_{00}^v \quad (95)$$

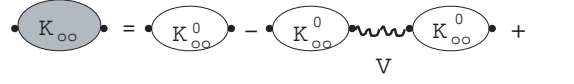


FIG. 3 The diagrammatic relation between the full response function  $K_{00}$  and the irreducible part  $K_{00}^v$  for a problem with just Coulomb interactions, denoted by the wiggly line  $V = v(q)$ . In RPA  $K_{00}^v$  is approximated by the particle-hole bubble  $K_{00}^0$ .

where we have used the continuity equation  $qj = \omega\rho$  and  $E = iq\phi$ .<sup>26</sup>

In the RPA  $K_{00}^v$  is approximated by  $K_{00}^0$ , the free particle-hole bubble. Thus RPA takes into account Coulomb interaction via the internal field the particle themselves generate, but ignores all vertex and self-energy corrections coming from exchanging  $v(q)$  inside the particle-hole bubble. Figure (3) shows the connection between the irreducible and total responses.

The HLR work differs from this illustrative example in that there are now two types of gauge fields, the Coulombic  $v(q)$  and the CS gauge field, so that the wiggly line in Fig. (3) is described by a  $2 \times 2$  matrix propagator. The HLR version of RPA consists of summing repeated bubbles irreducible with respect to both propagators, and approximating the irreducible part by the free-field response. Thus the CS field and  $v(q)$  are included only to take into account the internal fields the induced charges and currents produce. In matrix notation it is still true that

$$[K]^{-1} = [K^0]^{-1} + U \quad (96)$$

where  $U$  is a matrix propagator for the Coulombic  $v(q)$  and the CS gauge fields. Let us focus on just the 00 element  $K_{00}$ , which assumes the form

$$K_{00} = \frac{1}{v(q) + [K_{00}^0]^{-1} + \left(\frac{2\pi\bar{\phi}}{q}\right)^2 K_{11}^0} \quad (97)$$

where  $K_{11}^0$  is the free-field transverse-current response, and  $\bar{\phi}$  is the number of flux quanta attached. Though  $\bar{\phi} = 2s$  in our notation, we still use HLR's notation here to help readers who may wish to consult that work for more details.

Let us now examine this expression in various regimes.

### 1. Static compressibility

The behavior of  $K_{00}$  at  $\omega = 0$ ,  $q \rightarrow 0$ , gives the static compressibility of the system. For  $\omega = 0$  and  $q \ll k_F$ , they find

$$K_{00} = \frac{1}{v(q) + 2\pi(1 + \bar{\phi}^2/6)/m} \quad (98)$$

which shows that just as in a Fermi liquid, the compressibility is nonzero if  $v(q)$  is short ranged and vanishes as  $q$  if Coulombic. This vanishing only means that any applied external force is unable to change the density because the field inside the medium is strongly screened by the medium.

### 2. Cyclotron mode

The poles of  $K_{00}$  define the natural modes of oscillation of the system, for at these poles we get a response with no applied potential (see Eqn. (90)). If at any  $q$  there is a pole at some  $\omega(q)$ , it means that there is a mode of energy  $\omega(q)$ . For example, in a 3D electron gas, a plasmon pole appears at the plasma frequency as  $q \rightarrow 0$  and then moves as a function of  $q$ . At larger  $q$ , when decay into particle-hole pairs is possible, the pole position acquires an imaginary part denoting a finite lifetime.

In the present case, as  $q \rightarrow 0$ , at very high  $\omega$ ,  $K_{00}$  has a pole at the cyclotron frequency  $\omega_0 = 4\pi n/m$  (where  $m$  is the bare or band mass) with a residue in accordance with Kohn's theorem.

<sup>26</sup> The extra  $e$  in front comes in because  $\rho$  and  $j$  refer to the particle density and not charge density.

### 3. The overdamped mode.

At very low  $\omega$  and  $q$ , one finds

$$[K_{00}]^{-1} = \frac{2\pi}{m} \left(1 + \frac{\bar{\phi}^2}{12}\right) + v(q) - i \left(\frac{2\pi\bar{\phi}}{q}\right)^2 \frac{2n\omega}{qk_F}. \quad (99)$$

The zero occurs at the *overdamped mode* with the dispersion relation

$$\omega \simeq iq^3 v(q) \simeq q^2 \quad (100)$$

for the Coulomb case. If  $v(q)$  is short ranged, the mode is sub-diffusive. The reason charge diffuses rather than moving ballistically (even in the clean system) is that the magnetic field makes the charge move perpendicular to the Coulomb force, which tries to even out the density gradient. The overdamped mode and the effects it produces lie at the heart of the HLR work.

Note that if one first sends  $m \rightarrow 0$ , one would miss the over damped mode. Of course in a real system we can always find a  $q$  such that  $1/q$  dominates  $1/m$ . It is however a theoretical problem that we cannot send  $m \rightarrow 0$  here. A way out will be discussed later.

### 4. Longitudinal conductivity

Upon examining Eqn. (99), we can see that the Coulomb-irreducible part (obtained by dropping  $v$ ) is dominated by the last term in the limit of small  $q$  and  $\omega$ . Thus

$$\sigma_{xx} = e^2 \frac{\omega}{iq^2} K_{00}^v = \frac{e^2}{8\pi} \frac{q}{k_F}. \quad (101)$$

The above result holds only for  $q \gg 1/l_m$ , where  $l_m$  is the mean free path for the CF. For  $q \ll 1/l_m$

$$\sigma_{xx} = \frac{e^2}{4\pi k_F l_m}. \quad (102)$$

It is interesting to see how one arrives at Eqn. (102).

Let us define a CF conductivity by

$$e\mathbf{j} = \sigma_{CF} \mathbf{E}^T \quad (103)$$

where  $\mathbf{E}^T$  is the total field, which is the sum of the applied field and the internal field  $\mathbf{e}$  generated by the fermions themselves. To compute  $\mathbf{e}$  imagine a particle current  $\mathbf{j}$  and a unit length perpendicular to it. In one second  $j$  particles cross it carrying with them  $2j$  flux quanta of CS flux. However the CF makes no distinction between real flux and CS flux since they enter  $H_{CS}$  only via their sum. It will therefore sense a CS electric field  $\mathbf{e} = \frac{4\pi}{e} \hat{\mathbf{z}} \times \mathbf{j}$ . Thus

$$e\mathbf{j} = \sigma_{CF} \left[ \mathbf{E} + \frac{4\pi}{e} \hat{\mathbf{z}} \times \mathbf{j} \right]. \quad (104)$$

The electron resistivity tensor  $\rho$  (with components  $\rho_{\alpha\beta}$  and not to be confused with the density operator) defined by

$$E_\alpha = \rho_{\alpha\beta} (ej_\beta) \quad (105)$$

is then

$$\rho = \rho_{CS} + \rho_{CF} \quad (106)$$

where

$$\rho_{CS} = \frac{4\pi}{e^2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (107)$$

At mean-field level CF's do not see a field of any kind. The effect of disorder is to produce

$$\sigma_{CF, xx} = \sigma_{CF, yy} = \frac{ne^2\tau}{m} = \frac{ne^2l_m}{k_F} = \frac{k_F^2 e^2 l_m}{4\pi k_F} = \frac{e^2 k_F l_m}{4\pi} \quad (108)$$

where  $\tau$  is the elastic scattering time and  $l_m$  is the mean free path. If we now use Eqn. (106) and assume  $k_F l_m \gg 1$  we obtain Eqn. (102).

*Note that the resistivity of the electron is the sum of the resistivities of the CF and the CS term. If we move off to a general Jain fraction and ignore disorder, we will have*

$$\rho_{CF} = \frac{2\pi}{e^2} \begin{bmatrix} 0 & \frac{1}{p} \\ -\frac{1}{p} & 0 \end{bmatrix} \quad (109)$$

since the CF's fill  $p$  LL's of their own. When the resistivity matrices are added, one obtains the correct Hall resistivity for electrons at  $\nu = p/(2p+1)$ .

## 5. Surface acoustic waves (SAW)

When a surface acoustic wave is coupled to the electronic system, it is predicted to undergo a velocity shift and an attenuation described by

$$\frac{\delta v_s}{v_s} - \frac{i\kappa}{q} = \frac{\alpha^2/2}{1 + i\sigma_{xx}(q)/\sigma_m} \quad (110)$$

where  $\alpha$  is a piezoelectric constant,  $v_s$  is the sound velocity,  $\kappa$  describes the attenuation,  $\sigma_{xx}(q) = \sigma_{xx}(q, \omega = qv_s)$  where  $v_s$  is the sound velocity, and  $\sigma_m = \frac{v_s \epsilon}{2\pi}$ . Theory fits the experiments of Willett *et al* (1990,1996) (for a review see Willett (1997)) with a  $\sigma_m$  that is about five times larger than expected. The reader is strongly urged to consider HLR and the reviews for more details.

## 6. Mass divergences

HLR (1993) predicted a divergence in the effective mass  $m^*$  at the Fermi surface arising from the fermion self-energy diagram involving the emission and absorption of the overdamped mode. For the Coulomb case one has

$$m^*(\omega) \simeq \ln \omega. \quad (111)$$

(Shorter range interaction lead to more violent divergences.) Assuming this mass can be used near  $\nu = 1/2$  one expects that the gaps will be given by

$$E_\nu = \frac{eB^*}{m^*} \quad (112)$$

where  $m^*$  is self-consistently defined by  $m^*(\omega = E_\nu)$ . For  $\nu = p/(2p+1)$ , this implies, as  $p \rightarrow \infty$ , that  $E_p \simeq 1/(p \ln p)$ . The log divergent effective mass will also imply a specific heat  $C(T) \simeq T \ln T$ . It has not been possible to confirm these logarithms in numerical or experimental work.

Since the nature of the mass divergence depends on the range of the potential, it cannot be reproduced by trial wavefunctions, which make no explicit reference to the potential, in particular its range.

It has been shown that the mass divergences do not affect bosonic (e.g., density-density) correlations<sup>27</sup>.

## B. Physics near $\nu = 1/2$

The HLR (1993) predictions transcend  $\nu = 1/2$  and describe its immediate vicinity. The key idea is that in this region, it is useful to think of the CF as a particle seeing a weak effective magnetic field  $B^* = B/(2p+1)$ . This means that it will describe a cyclotron orbit of radius

$$R^* = \frac{\hbar k_F}{eB^*} \quad (113)$$

with  $k_F = \sqrt{4\pi n}$  (given that spin is fully polarized). Note that this result is independent of the fermion mass, whose treatment is quite tricky.

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<sup>27</sup> Kim, Furusaki, Wen, and Lee (1994), Kim, Lee, Wen, and Stamp (1994), Kim, Lee, and Wen (1995), Stern and Halperin (1995).

The SAW results of Willett *et al* (1990,1996) found that away from  $\nu = 1/2$ , there was a resonance in the velocity shift when the wavelength of the surface acoustic wave coincided with  $2R^*$ .

Confirmation of the  $R^*$  concept was also found in the experiments of Kang *et al* (1993), Goldman, Su, and Jain (1994), and Smet *et al* (1996). To visualize the Goldman *et al* experiment, imagine a semi-infinite system in the upper - half plane. If a current is introduced at the origin up the  $y$ -axis, it should bend and return to the  $x$ -axis at  $x = 2R^*$ , after completing one semi-circle. It would then bounce off and start the next semi-circle. It follows that if a return path is provided on the  $x$ -axis, the maximum current will flow if the drain is located an integral multiple of  $2R^*$  from the source. Or if a return location is held fixed, and  $B$  is varied, a maximum is expected and (found) whenever the spacing between source and drain is a multiple of  $2R^*$ .

Kang *et al* built an array of antidots, where each antidot is a region where electrons are absent. It was found that the sheet conductance had peaks when the antidot lattice constant equaled  $2R^*$ . While a detailed formula for conductance is not known in this context, the correlation is very suggestive and is what was seen with ordinary electrons in a weak field. The fact that CF's, which entered the theory as a mathematical device, manifest themselves so clearly in transport is a stunning affirmation of the theoretical framework.

### C. HLR - Room for improvement?

Despite its many remarkable and experimentally confirmed predictions, the HLR theory leaves room for improvement mainly because of the CS approach on which it is based.

One such area has to do with the dependence on bare mass  $m$ . We have argued that at the starting point  $H_{CS}$  must contain the bare mass since flux attachment by minimal coupling the CS field is an exact transformation done on electrons. This choice certainly helps to get the cyclotron frequency in accord with Kohn's theorem. However in our extraction of various low-energy long wavelength quantities like the over-damped mode from Eqn. (99), we had to assume that  $1/q$  dominates over  $1/m$ . While this is certainly valid for realistic and fixed values of  $m$ , one would like to be able to extract what is evidently the correct physics even if  $m = 0$  is imposed first.

Simon and Halperin (1993) and Simon, Stern, and Halperin (1996), proposed a way out. They assume, as in Landau theory, that  $H_{CS}$  is an effective theory with an effective  $m^*$ . Kohn's theorem and the over damped mode can both be salvaged if a suitable Landau parameter  $F_1$  is introduced. This is reasonable since Kohn's theorem relies on Galilean invariance and Landau's theory uses this principle to related  $m$ ,  $m^*$  and  $F_1$ . However this leaves the origin of  $m^*$  inside a black box.<sup>28</sup> It is also not clear why in the effective theory the fermion the CS field should be minimally coupled.

In the case of Coulomb interactions, one can also argue that the bare mass is swamped by the renormalized  $m^* \simeq \ln \omega$  generated by the exchange of the over-damped mode. Again it would be nice to be able to follow in detail the separation of the low energy physics controlled by an  $m^*$  generated by interactions and high-energy physics controlled by  $m$ .

Another shortcoming is that there is no evidence of the neutral fermion one expects at  $\nu = 1/2$ , and more generally a fermion of charge  $e^* = e/(2ps + 1)$  at other Jain fractions. This was to be expected since the CF in Halperin *et al* and in earlier CS work of Lopez and Fradkin was an electron bound to two flux tubes which carry no charge.

Also missing was the effective magnetic moment  $\mu^* = e/2m$  that Simon, Stern and Halperin (1996) would later argue must go with each CF.

In short, the CF that appears in the CS theory does not have, in obvious form, the right charge or energy scale of the ultimate quasiparticle. While the CS transformation is exact and can yield all these in principle, they are not manifest in the RPA.

Finally, Lee, Krotov, Gan, and Kivelson (1997, 1998) have raised the following issue. Suppose we push all LL's up to infinity except the LLL and assume particle-hole symmetric disorder at  $\nu = \frac{1}{2}$ . Then it can be verified that a nonzero  $\sigma_{xx}$  for electrons implies for CS-fermion a  $\sigma_{xy}^{CS} = -e^2/(4\pi)$ . However, both the mean-field approximation and the RPA corrections to it, give  $\sigma_{xy}^{CS} = 0$ . We are not aware of a resolution of this issue, arising from the fact that resistivities add.

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<sup>28</sup> Why do we view this inability to calculate  $m^*$  as a weakness, when in Landau theory it is simply accepted as a fact of life? As we shall see, in the FQHE we can go a long way towards computing  $m^*$  from the interactions by using the EHT.



## V. HAMILTONIAN THEORY II- EXTENDED HAMILTONIAN THEORY (EHT)

We now turn to our extension of the CS formalism<sup>29</sup>. In this extension, a CF with all the known properties can be made manifest and a variety of LLL quantities computed with no  $1/m$  singularities. But the computation of certain low energy long wavelength quantities (like the compressibility), straightforward in the CS approach, become extremely delicate. Hopefully we will impart to the reader a sense of which approach to use for what purpose.

We want to eschew the historical route and furnish an axiomatic presentation of our work with the minimum of preamble, setting the stage for exact or approximate calculations as rapidly as possible. However, to establish the context, we will begin with a brief sketch of our earlier work so as to take some of the mystery out of the end product and provide some degree of motivation.

Let us recall the work of Bohm and Pines (Bohm and Pines, 1953) on the electron gas in three dimensions. At small  $q$  the spectrum consists of particle-hole pairs at low energies and the plasmon at high energies. The particle and hole are part of the original hamiltonian while the plasmon comes from summing an infinite class of diagrams in the density-density response. It is not an independent entity, even though it is a sharply defined excitation that can be experimentally produced and detected. Bohm and Pines showed that by introducing extra canonical oscillator degrees of freedom at small  $q$ , one could describe plasmons as independent objects in an enlarged Hilbert space. In order to prevent double counting, they imposed constraints on state vectors of the form

$$\bar{\chi}(\mathbf{q})|\text{physical state}\rangle = 0 \quad (114)$$

with one  $\bar{\chi}(\mathbf{q})$  for each  $\mathbf{q}$  at which a plasma mode was introduced as an independent canonical oscillator. The plasmons, initially coupled to the fermions, were approximately decoupled, leaving behind fermions with a renormalized mass, and constraints that essentially froze out any putative plasmons with small  $q$ . Such a simple description of plasmons, or isolation of high-energy physics, would have been impossible within the confines of the electronic Hilbert space, wherein plasmons are complicated collective excitations of the electrons themselves.

What we originally did for the FQHE (Shankar and Murthy, 1997, Murthy and Shankar, 1998a) was similar in many ways. We enlarged the Hilbert space to include at each  $\mathbf{q}$ , a new set of independent canonically conjugate variables - a transverse vector field  $\mathbf{a}(\mathbf{q})$  and a longitudinal vector field  $\mathbf{P}(\mathbf{q})$ . Using these it was possible, by a unitary transformation, to get rid of the dependent CS vector potential  $\mathbf{a}_{CS}$ . While all the operators in our hamiltonian were now independent, the physical sector was defined by the (CS) condition

$$(\nabla \times \mathbf{a} - 4\pi s\rho)|\text{physical state}\rangle = 0. \quad (115)$$

The conjugate variables  $\mathbf{a}$  and  $\mathbf{P}$  formed oscillators near, but not exactly at, the cyclotron frequency. They were coupled to fermions. We found a way to decouple the oscillators in the limit  $ql \rightarrow 0$  by a second unitary transformation. The decoupled oscillators now complied with Kohn's theorem for pole position and residue. The formula for  $\bar{\rho}(\mathbf{q})$ , the electronic charge density projected to the LLL, was derived to order  $ql$ , as was the constraint  $\bar{\chi}(\mathbf{q})$  which now acted only within the particle sector. At this point a guess was made (Shankar 1999) to extend the small  $ql$  answer to all orders by exponentiation of leading-order terms. The final theory of the LLL sector took the following form:

$$\bar{H} = \frac{1}{2} \sum_{\mathbf{q}} \bar{\rho}(\mathbf{q}) v(q) e^{-q^2 l^2 / 2} \bar{\rho}(-\mathbf{q}) \quad (116)$$

$$[\bar{H}, \bar{\chi}(\mathbf{q})] = 0 \quad (117)$$

$$0 = \bar{\chi}(\mathbf{q})|\text{physical state}\rangle \quad (118)$$

The hamiltonian  $\bar{H}$  is just the potential energy  $V$  projected to the LLL, written in terms of the projected electron charge density  $\bar{\rho}(\mathbf{q})$

$$\bar{\rho}(\mathbf{q}) = \sum_j \exp(-i\mathbf{q} \cdot \mathbf{R}_{ej}) \quad (119)$$

$$\mathbf{R}_e = \mathbf{r} - \frac{l^2}{1+c} \hat{\mathbf{z}} \times \boldsymbol{\Pi} \quad (120)$$

where we define a very important and frequently occurring variable:

$$c^2 = \frac{2ps}{2ps+1}. \quad (121)$$

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<sup>29</sup> Murthy and Shankar (2002), Shankar and Murthy (1997), Murthy and Shankar (1998a,b, 1999, 2002), Murthy (1999, 2000a,c, 2001b,c), Shankar (1999, 2000, 2001), Murthy, Park, Shankar, and Jain (1998).

From the commutation relations

$$[\mathbf{R}_{ex}, \mathbf{R}_{ey}] = -il^2 \quad (122)$$

it is clear that  $\mathbf{R}_e$  is just the electron guiding center coordinate *but now expressed in terms of CF variables*  $\mathbf{r}$  and  $\mathbf{\Pi} = \mathbf{p} + e\mathbf{A}^*$ .

The physical states are annihilated by the constraint

$$\bar{\chi}(\mathbf{q}) = \sum_j \exp(-i\mathbf{q} \cdot \mathbf{R}_{vj}) \quad (123)$$

$$\mathbf{R}_v = \mathbf{r} + \frac{l^2}{c(1+c)} \hat{\mathbf{z}} \times \mathbf{\Pi} \quad (124)$$

The *pseudovortex coordinate*  $\mathbf{R}_v$  describes an object of charge  $-c^2 = -2ps/(2ps+1)$ :

$$[\mathbf{R}_{vx}, \mathbf{R}_{vy}] = \frac{il^2}{c^2} \quad (125)$$

and commutes with  $\mathbf{R}_e$ :

$$[\mathbf{R}_e, \mathbf{R}_v] = 0. \quad (126)$$

Thus the constraints commute with  $\bar{H}$ , and form an algebra:

$$[\bar{\chi}(\mathbf{q}), \bar{\chi}(\mathbf{q}')] = -2i \sin \left[ \frac{l^2(\mathbf{q} \times \mathbf{q}')}{2c^2} \right] \bar{\chi}(\mathbf{q} + \mathbf{q}'). \quad (127)$$

The problem is just like Yang-Mills theory.

The expressions for  $\mathbf{R}_e$  and  $\mathbf{R}_v$  in terms of  $\mathbf{r}$  and  $\mathbf{\Pi}$  which were already encoded in the small- $\mathbf{q}$  theory, jumped out upon exponentiation. They, together with the constraints, lie at the heart of our approach.

*We will now show how one can get to the final result, Equations (116-118), if we simply make a certain enlargement of the electronic Hilbert space and follow it with a change of variables.* Any reader who wants to know more about what makes us introduce these variables should consult our earlier work. Those who just want to use the results can go ahead.

### A. The axiomatic introduction to the EHT

Let us begin afresh with the primordial hamiltonian in terms of electronic variables (which carry the subscript  $e$  to distinguish them from other coordinates to be introduced):

$$H = \sum_j \frac{\eta_{ei}^2}{2ml^4} + \frac{1}{2} \sum_{i,j,\mathbf{q}} v(q) e^{i\mathbf{q} \cdot (\mathbf{r}_{ei} - \mathbf{r}_{ej})} \equiv H_0 + V. \quad (128)$$

This hamiltonian contains complete information about the problem. It can be used to study LL-mixing and the computation of Hall current which requires higher LL's in an essential way. These topics will be discussed in due course. We first focus on the main challenge of hamiltonian theories: extracting the  $m$ -independent physics of the LLL. As discussed in Section (II), projecting to the LLL as such is no problem: one drops the first term and makes the replacement  $\mathbf{r} \rightarrow \mathbf{R}$  in the density operator and  $v(q) \rightarrow v(q)e^{-q^2 l^2/2}$ . The catch is that the projected hamiltonian lives in the highly degenerate LLL, frustrating both perturbation theory and the Hartree-Fock approximation.

In the CS approach one resorts to flux attachment to beat the degeneracy of the kinetic term, but that can be done only in the full electronic Hilbert space. Consequently  $m$  gets into everything and the low (LLL) and high-energy sectors get hopelessly entangled. What we really want to do is work within the LLL and attach flux tubes, which in the LLL translates into vortices, by analyticity. However when we say vortices, we do not mean zeros of the wave function, for such a thing does not exist as a degree of freedom within the hamiltonian and as we have seen, there are not enough of them in the wavefunction to go around anyway. *Instead we mean by a vortex some object which has the charge of the  $2s$ -fold vortex and corresponds to an excitation that can be created by inserting  $2s$  flux quanta into the Hall system.* Since such an object does not exist in the original Hilbert space (as an independent entity) we enlarge it to make room for this entity, which we call the *pseudovortex*, the *vortex* to emphasize its similarity to the vortices in the wavefunctions, and the *pseudo* to emphasize its differences.

The enlargement of Hilbert space can be explained in terms of just one electron, with cyclotron and guiding center coordinates  $\boldsymbol{\eta}_e$  and  $\mathbf{R}_e$ . Let us temporarily focus on just the LLL physics and ignore  $\boldsymbol{\eta}_e$  which does not participate in the change of variables, and will be reinstated subsequently.

First we introduce an extra guiding center coordinate  $\mathbf{R}_v$  (the pseudovortex), defining it algebraically by its commutation relations which represent a charge  $-c^2$

$$[R_{vx}, R_{vy}] = \frac{il^2}{c^2}. \quad (129)$$

Next we combine  $\mathbf{R}_e$  and  $\mathbf{R}_v$  to form the CF space. (Note that it takes two canonical pairs to make one regular fermion in two dimensions.) This CF space can be defined in terms of either the position  $\mathbf{r}$  and velocity  $\boldsymbol{\Pi}$  of the CF, or in terms of its cyclotron and guiding center coordinates,  $\boldsymbol{\eta}$  and  $\mathbf{R}$ . **Note that henceforth variables carrying not indentifying subscript will refer to the CF.** The CF coordinates  $\boldsymbol{\eta}$  and  $\mathbf{R}$  obey the commutation rules of the cyclotron and guiding center coordinates of an object of charge  $e^* = 1 - c^2 = 1/(2ps + 1)$ :

$$[\eta_x, \eta_y] = il^{*2} = \frac{il^2}{1 - c^2} \quad (130)$$

$$[R_x, R_y] = -il^{*2}. \quad (131)$$

The rule for forming the CF from  $\mathbf{R}_e$  and  $\mathbf{R}_v$  is the following:

$$\mathbf{R} = \frac{\mathbf{R}_e - c^2 \mathbf{R}_v}{1 - c^2} \quad (132)$$

$$\boldsymbol{\eta} = \frac{c}{1 - c^2} (\mathbf{R}_v - \mathbf{R}_e) \quad (133)$$

The first equation says that the CF guiding center is the weighted sum of its parts. The second can be found by demanding that  $\boldsymbol{\eta}$  be linear in  $\mathbf{R}_e$  and  $\mathbf{R}_v$ , commute with  $\mathbf{R}$ , and have an overall scale that produces the right commutator.

The inverse transformation is

$$\mathbf{R}_e = \mathbf{R} + \boldsymbol{\eta}c \quad (134)$$

$$\mathbf{R}_v = \mathbf{R} + \boldsymbol{\eta}/c \quad (135)$$

In terms of  $\mathbf{r}$  and  $\boldsymbol{\Pi}$ , the CF coordinate and velocity operators

$$\mathbf{R}_e = \mathbf{r} - \frac{l^2}{(1 + c)} \hat{\mathbf{z}} \times \boldsymbol{\Pi}, \quad (136)$$

$$\mathbf{R}_v = \mathbf{r} + \frac{l^2}{c(1 + c)} \hat{\mathbf{z}} \times \boldsymbol{\Pi} \quad (137)$$

which are just the expressions encountered in the brief historical review.

Ignoring the zero point energy, here is where we stand in the LLL sector:

$$\bar{\bar{H}} = \frac{1}{2} \sum_{i,j,\mathbf{q}} v(q) e^{-q^2 l^2 / 2} e^{i\mathbf{q} \cdot (\mathbf{R}_{ei} - \mathbf{R}_{ej})} \quad (138)$$

$$= \frac{1}{2} \sum_{i,j,\mathbf{q}} v(q) e^{-q^2 l^2 / 2} \exp(i\mathbf{q} \cdot [(\mathbf{R}_i - \mathbf{R}_j) + c(\boldsymbol{\eta}_i - \boldsymbol{\eta}_j)]) \quad (139)$$

While it is true that we have managed to get rid of  $m$  and isolate the LLL cleanly, the reader may ask what we have gained, since *algebraically* the problem is the same as in electronic coordinates. *The answer is that now there is a natural nondegenerate HF ground state in the extended space.* This is because the HF hamiltonian is now written in terms of CF operators  $\mathbf{R}$  and  $\boldsymbol{\eta}$  and the particle density is just right to fill exactly  $p$ -filled CF-LL's, i.e.,  $\chi_p$  is the ground state. The proof of its HF nature is found in Appendix C. This key step opens up all the usual approximation schemes.

Depending on what we want to compute, there are two distinct schemes. Both rely on the nondegenerate HF ground state, and both acknowledge a huge symmetry group of  $\bar{\bar{H}}$ , which comes from the following fact:  $\bar{\bar{H}}$ , as embedded in the CF space, does not depend on  $\mathbf{R}_v = \mathbf{R} + \boldsymbol{\eta}/c$ , the pseudovortex coordinate. Equivalently, it depends on  $\mathbf{R}$  and

$\boldsymbol{\eta}$  only through the combination  $\mathbf{R}_e = \mathbf{R} + c\boldsymbol{\eta}$ . The other (commuting) combination  $\mathbf{R}_v = \mathbf{R} + \boldsymbol{\eta}/c$  can be used to define a *pseudovortex density*

$$\bar{\chi}(\mathbf{q}) = \sum_j e^{-i\mathbf{q}\cdot\mathbf{R}_v} \quad (140)$$

which commutes with  $\bar{H}$ :

$$[\bar{H}, \bar{\chi}(\mathbf{q})] = 0. \quad (141)$$

Let us understand this symmetry. If we view  $\bar{H}$  as a function of  $\boldsymbol{\eta}$  and  $\mathbf{R}$ , or  $\mathbf{r}$  and  $\boldsymbol{\Pi}$ , its eigenfunctions will depend on two coordinates, which can be chosen to be one component each of  $\boldsymbol{\eta}$  and  $\mathbf{R}$  or just  $x$  and  $y$ . If we view  $H$  as a function of  $\mathbf{R}_e$  and  $\mathbf{R}_v$ , it depends only on  $\mathbf{R}_e$ . The energy eigenfunctions will be of the form

$$\psi(z_e, z_v) = \psi_e(z_e)\psi_v(z_v) \quad (142)$$

where  $\psi_v(z_v)$  is arbitrary since nothing in  $\bar{H}$  determines it. This degeneracy is of the same type as that of the noninteracting electron hamiltonian which depends on  $\boldsymbol{\eta}_e$  but not  $\mathbf{R}_e$ . However, since  $\mathbf{R}_v$  is unphysical, and all physical observables depend only on  $\mathbf{R}_e$ , this is a gauge symmetry.

We deal with the gauge symmetry as usual, by selecting a representative from each orbit, which in this case means eliminating the degeneracy due to the arbitrary function  $\psi(z_v)$  that tags along for the ride. Let us make a particular choice  $\psi_{v0}(z_v)$ . All we require is that it be translationally invariant so that  $\langle \bar{\chi}(\mathbf{q}) \rangle = 0$  in this state. The gauge-fixed Hilbert space now consists of functions of the form  $\psi_e(z_e)\psi_{v0}(z_v)$ . In this sector  $\bar{\chi}(\mathbf{q}) = 0$  in the weak sense: any Green's function involving a string of  $\bar{\chi}(\mathbf{q})$ 's will vanish since (i)  $\bar{\chi}(\mathbf{q}, t) = \bar{\chi}(\mathbf{q}, 0) \equiv \bar{\chi}(\mathbf{q})$ , (ii)  $\langle \bar{\chi}(\mathbf{q}) \rangle = 0$ ,<sup>30</sup> in the one-dimensional space spanned by  $\psi_{v0}(z_v)$ . (Imagine inserting the projector to this state between any two  $\bar{\chi}(\mathbf{q})$ 's in the Green's function.)

In the path integral language we can make  $\bar{\chi}(\mathbf{q})$  vanish *weakly*, that is to say, vanish whenever it appears in a Green's function, by a method similar to what is done in gauge theories. Imagine writing a path integral in the full CF space and then inserting a delta function imposing  $\bar{\chi}(\mathbf{q}) = 0$  for all  $\mathbf{q}$  at any one time.<sup>31</sup> Since  $\bar{\chi}(\mathbf{q})$  does not change with time, this restriction will hold automatically for all times and specify the fate of  $\bar{\chi}(\mathbf{q})$  the way a gauge fixing terms does for the longitudinal degrees of freedom.

The theory is thus defined (in schematic form) by the equations

$$\bar{H} = \frac{1}{2} \sum_{\mathbf{q}} v(q) e^{-q^2 t^2 / 2} \bar{\rho}(\mathbf{q}) \bar{\rho}(-\mathbf{q}) \quad (143)$$

$$[\bar{H}, \bar{\chi}(\mathbf{q})] = 0 \quad (144)$$

$$\bar{\chi}(\mathbf{q}) \simeq 0 \quad (145)$$

where  $\simeq 0$  means vanish weakly.

Now we turn to the two approximate ways of dealing with this problem: the conserving approximation and the shortcut.

## B. The conserving approximation

Given a hamiltonian, a reasonable first approximation to try is Hartree-Fock. It is shown in Appendix C that CF-LL states are Hartree-Fock states of  $\bar{H}$ . How good is this Hartree-Fock approximation likely to be? For example, how will we fare if we compute the activation or *transport gap* in a fully polarized sample as per:

$$\Delta = \langle \mathbf{p} + PH | H | \mathbf{p} + PH \rangle - \langle \mathbf{p} | H | \mathbf{p} \rangle \quad (146)$$

where  $|\mathbf{p}\rangle$  stands for the Hartree-Fock ground state with  $p$ -filled LL's and  $PH$  stands for a widely separated particle-hole pair?

There are at least two good reasons to expect that this naive HF result will require fairly strong corrections. First, if we compute the matrix element of the projected electron density between any two HF states, the answer will be

<sup>30</sup> An exception occurs if the string contains only  $\bar{\chi}(0) = N$ . This does not affect what we plan to do.

<sup>31</sup> Since  $\bar{\chi}(\mathbf{q})$  involves  $\mathbf{r}$  and  $\mathbf{p}$ , this will have to be done in the phase space path integral.

linear in  $q$ , whereas in the exact theory, and within the LLL, it must go as  $q^2$  as per Kohn's theorem. To see this note that

$$e^{-i\mathbf{q}\cdot\mathbf{R}_e} = 1 - i\mathbf{q}\cdot(\mathbf{R} + c\boldsymbol{\eta}) + \mathcal{O}(q^2). \quad (147)$$

While  $\mathbf{R}$  has no transition matrix elements between different CF Landau levels,  $\boldsymbol{\eta}$  does. The second problem is that as  $ql \rightarrow 0$ , the projected electronic density (which reduces to  $\sum_j \exp(-i\mathbf{q}\cdot\mathbf{r}_j)$ ) has unit contribution from each CF while we would like it to be  $e^*$ . Evidently the HF result will receive strong corrections that will renormalize these quantities till they are in line with these expectations.

These shortcomings are to be expected since the HF solution does not obey the constraint, or equivalently, does not factorize into the form  $\psi = \psi_e(z_e)\psi_v(z_v)$ .

The conserving approximation (Anderson, 1958a,1958b, Rickayzen, 1958, Nambu, 1960, Baym and Kadanoff, 1961) is a sophisticated procedure for improving the HF state with additional diagrammatic corrections so that  $\bar{\chi}(\mathbf{q}) \simeq 0$  in Green's functions. For  $\nu = 1/2$  Read (1998) showed that this procedure restores Kohn's theorem, and reveals a dipolar structure for density-density correlations. We shall say more about this in connection with the compressibility paradox.

Now we discuss the other approximation, the shortcut.

### C. The shortcut: The preferred charge and hamiltonian

Consider the exact solution to the gauge fixed problem. Suppose, in the hamiltonian and elsewhere, we replace  $\bar{\rho}(\mathbf{q})$  by the *preferred combination*

$$\bar{\rho}^p(\mathbf{q}) = \bar{\rho} - c^2\bar{\chi}. \quad (148)$$

This makes no difference (to the computation of anything physical) in an exact calculation, since  $\bar{\chi}$  is essentially zero.

However, in the HF approximation it makes a big difference if we start with the hamiltonian written in terms of  $\bar{\rho}^p(\mathbf{q})$ . To see why, consider its expansion in powers of  $ql$ :

$$\bar{\rho}^p = \sum_j e^{-i\mathbf{q}\cdot\mathbf{r}_j} \left( \frac{1}{2ps+1} - il^2\mathbf{q} \times \boldsymbol{\Pi}_j + 0 \cdot (\mathbf{q} \times \boldsymbol{\Pi}_j)^2 + \dots \right). \quad (149)$$

- The transition matrix elements are of order  $q^2$  between HF states because coefficient of  $\mathbf{q}$  is proportional to the CF guiding center coordinate  $\mathbf{r} - l^{*2}\dot{\mathbf{z}} \times \boldsymbol{\Pi}$  with no admixture of the CF cyclotron coordinate. This is more transparent if we use  $\mathbf{R}$  and  $\boldsymbol{\eta}$  to write

$$\bar{\rho}^p(\mathbf{q}) = (1 - i\mathbf{q}\cdot(\mathbf{R} + c\boldsymbol{\eta}) + \dots) - c^2(1 - i\mathbf{q}\cdot(\mathbf{R} + \boldsymbol{\eta}/c + \dots)) \quad (150)$$

$$= (1 - c^2)(1 - \mathbf{q}\cdot\mathbf{R} + \mathcal{O}(q^2)). \quad (151)$$

- With no further fixing, we see that the electronic charge density associated with  $\bar{\rho}^p(\mathbf{q})$  is now  $1 - c^2 = e^*$ .
- We see from Eqn. (149) that when  $\nu = 1/2$ , the preferred density couples to an external electric field like a dipole of size  $l^{*2}\dot{\mathbf{z}} \times \mathbf{p}$ .

Thus there is not *a priori* need for strong corrections, at least in the long-wavelength limit.

The hamiltonian we work with

$$\bar{H}^p = \frac{1}{2} \sum_{\mathbf{q}} \bar{\rho}^p(\mathbf{q}) v(q) e^{-q^2 l^2/2} \bar{\rho}^p(-\mathbf{q}) \quad (152)$$

subsumes a lot of the right low-energy physics and CF properties. Unlike  $\bar{H}$  which commutes with  $\bar{\chi}(\mathbf{q})$ ,  $\bar{H}^p$  is *weakly gauge invariant*, that is

$$[H(\bar{\rho}^p), \bar{\chi}(\mathbf{q})] \simeq 0 \quad (153)$$

where the  $\simeq 0$  symbol means that it vanishes in the subspace obeying  $\bar{\chi}(\mathbf{q}) = 0$ . Thus neither  $H(\bar{\rho}^p)$  nor  $\bar{\rho}^p$  will mix physical and unphysical states.

The significance of  $H(\bar{\rho}^p)$  is the following. If the constraint  $\bar{\chi} = 0$  is imposed *exactly*, there are many equivalent hamiltonians depending on how  $\bar{\chi}$  is insinuated into it. However, in the HF *approximation*, these are not equivalent and  $H(\bar{\rho}^p)$  best approximates, between HF states and at long wavelengths, the true hamiltonian between true eigenstates. In contrast to a variational calculation where one searches among states for an optimal one, here the HF states are

the same for a class of hamiltonians (where  $\bar{\chi}$  is introduced into  $H$  in any rotationally invariant form), and we seek the best hamiltonian:  $H(\bar{\rho}^p)$  encodes the fact that every electron is accompanied by a correlation hole of some sort which leads to a certain  $e^*$ ,  $d^*$  and obeys Kohn's theorem ( $q^2$  matrix element for the LLL projected charge density).

Note that when we use the preferred charge and hamiltonian we will make no further reference to constraints, and simply carry out the Hartree-Fock approximation. This is based on the expectation that even if we found some way to include the effect of constraints, it will make no difference in the small  $ql$  region. This is because the leading renormalization of  $e$  to  $e^*$  and suppression of  $q$  matrix elements down to  $q^2$  that are achieved by the conserving approximation by summing ladder diagrams following Read (1998) or using the time-dependent HF (Murthy, 2001a, Section VI) are built in here.

*It is in this approximate, operator sense, where we use  $\bar{\rho}^p$  in place of  $\bar{\rho}$  that the binding of electrons and pseudovortices to form CF's is realized in the Hamiltonian theory.* Since the pseudovortices have coordinates that are independent of the electrons, there is no double-counting here. The problems that we encountered with vortices in the wavefunction approach for non-Laughlin fractions (with the limiting case being the dipole picture of  $\nu = \frac{1}{2}$ ) are also absent since we are not talking about those vortices. Since the pseudovortices *per electron* are independent of electrons, their number does not change when  $\nu$  changes (though their charge, tied to  $c^2 = 2s\nu$ , does.) Whereas antisymmetrization fragmented the vortices in the Jastrow factor into ordinary parametric zeros (except for the Pauli zero), it does nothing to the pseudovortices. Antisymmetrization is accomplished by simply writing the operator  $\bar{\rho}(\mathbf{q})$  in terms of second quantized (composite) fermion operators.

The reader will recall that any *simple* picture of quasiparticles, whether it be in Landau's Fermi liquid theory, or in BCS theory, is best captured by approximate and not exact descriptions. The quasiparticles are all caricatures of some exact reality and therein lies their utility. Similarly the CF in our extended formalism appears only in the HF approximation to  $\bar{H}^p$ . Recall that we brought in the coordinate  $\mathbf{R}_v$  to become the electron's partner in forming the CF. However  $\mathbf{R}_v$  was cyclic in the exact hamiltonian  $\bar{H}$ . *Thus the exact dynamics never demanded that  $\mathbf{R}_v$  be bound to  $\mathbf{R}_e$  or even be anywhere near  $\mathbf{R}_e$ .* However, in the HF approximation, since we wanted the right charge and transition matrix elements of the density operator (Kohn's theorem) to be manifest, we needed to replace  $\bar{\rho}(\mathbf{q})$  by  $\bar{\rho}^p(\mathbf{q})$ , and trade  $\bar{H}$  for  $\bar{H}^p$ , the preferred hamiltonian. In  $\bar{H}^p$ ,  $\mathbf{R}_v$  is coupled to  $\mathbf{R}_e$ . The HF approximation and this coupling go hand in hand. The exact eigenfunctions of the original  $\bar{H}$  are factorized in the analytic coordinates  $z_e$  and  $z_v$  and presumably reproduce the electronic correlations of the FQHE states. On the other hand, in the HF approximation to  $\bar{H}^p$ , the wavefunctions (e.g.,  $p$ -filled LL's) mix up  $z_e$  and  $z_v$ , and  $\bar{H}^p$ , the preferred hamiltonian, dynamically couples  $\mathbf{R}_e$  and  $\mathbf{R}_v$ . The net result is that, at least at long wavelengths, these two wrongs make it right and mimic what happens in the exact solution.

Another advantage of  $\bar{H}^p$  is that it gives an approximate formula for  $m^*$  originating entirely from interactions. This is best seen at  $\nu = 1/2$ . When we square  $\bar{\rho}^p$  (Eqn. (149)), we get a double sum over particles whose diagonal part is the one particle (free-field) term:

$$H_{\nu=\frac{1}{2}}^0 = 2 \sum_j \int \frac{d^2 q}{4\pi^2} \sin^2 \left[ \frac{\mathbf{q} \times \mathbf{k}_j l^2}{2} \right] v(q) e^{-q^2 l^2 / 2}. \quad (154)$$

This is not a hamiltonian of the form  $k^2/2m^*$ . However if the potential is peaked at very small  $q$ , we can expand the sine and read off an approximate  $1/m^*$

$$\frac{1}{m^*} = \int \frac{q dq d\theta}{4\pi^2} [(\sin^2 \theta) (ql)^2] v(q) e^{-q^2 l^2 / 2} \quad (155)$$

which has its origin in electron-electron interactions. However we can do more: we have the full  $H_0$  as well as the interactions. The point to emphasize is that  $H$  is not of the traditional form  $(p^2/2m + V)$  and that there is no reason it had to be.

The reader should verify that if we use  $H(\bar{\rho})$  instead, the one-particle piece will be a constant with no momentum dependence. The entire energy will be due to the Fock term, as in Read's (1998) conserving calculation.

For the benefit of readers who may be overwhelmed by seeing too many approaches, we present the key equations of the CS and EHT approaches in Table I.

To summarize, in any context where  $\bar{\rho}^p$  can be reliably employed, we can say that the CF appears to be the bound state of an electron and the pseudovortex. We shall see that this includes calculations of the gap and magnetization properties at zero and nonzero temperatures, even at  $\nu = 1/2$ . We shall turn to these later on. But first, we list some cases where  $\bar{\rho}(\mathbf{q})^p$  and  $\bar{H}^p$  fail to capture the right physics.

TABLE I Basic equations

|  |  |  |
|--|--|--|
|  | <p style="text-align: center;"><b>CS Theory</b></p> $H_{CS} = \frac{1}{2m} \sum_i (\mathbf{p} + e\mathbf{A} + \mathbf{a}_{cs})_i^2 + V$ $\nabla \times \mathbf{a}_{cs} = 4\pi s\rho$ <p style="text-align: center;"><b>Extended hamiltonian Theory (LLL only)</b></p> $\bar{H} = \frac{1}{2} \sum_{\mathbf{q}} v(q) e^{-q^2 l^2 / 2} \bar{\rho}(\mathbf{q}) \bar{\rho}(-\mathbf{q})$ $[\bar{H}, \bar{\chi}(\mathbf{q})] = 0$ $\bar{\chi}(\mathbf{q}) \simeq 0$ $\bar{\rho}(\mathbf{q}) = \sum_j \exp(-i\mathbf{q} \cdot \mathbf{R}_{ej})$ $\mathbf{R}_e = \mathbf{r} - \frac{l^2}{1+c} \hat{\mathbf{z}} \times \boldsymbol{\Pi} = \mathbf{R} + \boldsymbol{\eta} c$ $\bar{\chi}(\mathbf{q}) = \sum_j \exp(-i\mathbf{q} \cdot \mathbf{R}_{vj})$ $\mathbf{R}_v = \mathbf{r} + \frac{l^2}{c(1+c)} \hat{\mathbf{z}} \times \boldsymbol{\Pi} = \mathbf{R} + \boldsymbol{\eta}/c$ $\bar{H}^p = \bar{H}(\bar{\rho}^p)$ $\bar{\rho}^p(\mathbf{q}) = \bar{\rho}(\mathbf{q}) - c^2 \bar{\chi}(\mathbf{q})$ |  |
|--|--|--|

#### D. The conserving approximation and compressibility paradox

HLR predicted that the  $\nu = 1/2$  system has a static compressibility that is nonzero for short range forces and vanishes as  $q$  for the Coulomb interaction. This result appears paradoxical in the present approach in which the dipolar nature of the CF has been transparently exposed (using  $\bar{\rho}^p$ ). Imagine coupling the system to an external potential  $\Phi$ . The dipole will couple to the *gradient* of  $\Phi$  and the resulting response will be a dipolar density whose *divergence* will give the induced charge. These two italicized factors imply a  $q^2$  in the response even for short range interactions. Indeed, this is what we first obtained (Shankar and Murthy, 1997, Murthy and Shankar, 1998a) upon doing a simple RPA calculation. How is this to be reconciled with the HLR (1993) result, assuming their (compressible) answer is right?

Halperin and Stern (1998) first raised this question and provided the key to its resolution. They first established a matter of principle, namely, that dipolar objects could be compressible, by considering the following hamiltonian:

$$H = \sum_i \frac{\mathbf{p}_i^2}{2m} - \frac{1}{2mn} \sum_{i,j,\mathbf{q}}^{n,n,Q} \mathbf{p}_i \cdot \mathbf{p}_j e^{i\mathbf{q} \cdot (\mathbf{r}_i - \mathbf{r}_j)}. \quad (156)$$

This hamiltonian arose in our earlier work (Murthy and Shankar, 1998a) when we decoupled the magnetoplasmon oscillators from the fermions. There we had chosen the upper cut-off  $Q = k_F$  so that the  $i = j$  term from the double sum contributes  $-\mathbf{p}^2/2m$  to each particle and cancels the first sum, rendering  $1/m^* = 0$ . Halperin and Stern considered the limiting case  $Q \rightarrow 0$ . In this limit  $H$  takes the form

$$H = \sum_{ij} \frac{(\mathbf{p}_i - \mathbf{p}_j)^2}{2m} = \sum_i \frac{(\mathbf{p}_i - \frac{1}{n} \sum_j \mathbf{p}_j)^2}{2m} \quad (157)$$

and is invariant under the simultaneous shift of all momenta. This symmetry called *K-invariance* had also been pointed out by Haldane (1995) in unpublished work and arose as part of a gauge symmetry in our work. The symmetry implies that it costs no energy to move the Fermi surface as a whole. Consequently there are some very soft modes that could lead to a singular density response which can offset the  $q^2$  from the dipolar factors, provided these soft modes are not merely gauge artifacts that couple to nothing physical. The detailed calculation of Stern, Halperin, von Oppen and Simon (1999) (hereafter Stern *et al*) which we now describe, demonstrated that gauge invariant soft modes do exist and lead to nonzero compressibility.

The first order of business for them was to start with a hamiltonian that had *K*-invariance for small  $Q$  and not just  $Q = 0$ , since one needed to consider the response functions at small but nonzero  $q$ . The hamiltonian in Eqn. (157) had to be augmented by more terms to assure this. These terms were derived by Stern *et al* (1999) as follows. They go a step back in our calculation and start with the following hamiltonian and constraints for the coupled oscillators and particles:

$$\begin{aligned}
H &= \sum_j \frac{(\mathbf{p}_j - \mathbf{a})^2}{2m} + V \\
&= \sum_j \frac{\mathbf{p}_j^2}{2m} - \sum_{\mathbf{q}}^Q \mathbf{g}(\mathbf{q}) \frac{1}{2mn_{CS}} \mathbf{a}(-\mathbf{q})
\end{aligned} \quad (158)$$

$$+ \int d^2r [n_{CS}(\mathbf{r})|\mathbf{a}(\mathbf{r})|^2] + V \quad (159)$$

$$\mathbf{g}(\mathbf{q}) = \sum_j \mathbf{p}_j e^{-i\mathbf{q} \cdot \mathbf{r}_j} \quad (160)$$

$$0 = (\nabla \times \mathbf{a} - 4\pi n_{CS}) |\text{physical state}\rangle \quad (161)$$

where  $n_{CS}$  is the fermion number density operator. The only minor difference from our work is that  $a_x$  and  $a_y$  (instead of the longitudinal and transverse components  $a$  and  $P$ ) are canonically conjugate. The cut-off on  $q$  at  $Q$  means the Chern-Simons fermions now carry flux tubes, ones smeared over a distance  $1/Q$  (sometimes called “fat” flux tubes, see Halperin, 1992).

Next Stern *et al* (1999) approximate  $n_{CS}$  by  $n$ , the average density and obtain

$$H = \sum_j \frac{\mathbf{p}_j^2}{2m} - \frac{1}{2mn} \sum_{\mathbf{q}} \mathbf{g}(\mathbf{q}) \mathbf{g}(-\mathbf{q}) + \sum_{\mathbf{q}} \frac{n}{2m} \left| \mathbf{a}(\mathbf{q}) - \frac{1}{n} \mathbf{g}(\mathbf{q}) \right|^2 \quad (162)$$

Now they invoke the unitary transformation we employed to decouple the  $\mathbf{a}$  fields from the fermions in the small  $ql$  limit:

$$U = \exp \left[ \frac{i}{4\pi n} \sum_{\mathbf{q}} \mathbf{g}(\mathbf{q}) \times \mathbf{a}(-\mathbf{q}) \right] \quad (163)$$

Since  $a_x$  and  $a_y$  are conjugates,  $U$  is just the shift operator  $\mathbf{a} - \mathbf{g}/n \rightarrow \mathbf{a}'$ . Here comes the big difference. While we kept just the  $H$  from Eqn. (157) they augment  $H$  with additional terms to ensure that the constraints

$$\rho(\mathbf{q}) = -\frac{il^2}{2} \mathbf{q} \times \mathbf{g}(\mathbf{q}) \quad (164)$$

commute with  $H_{aug}$  to the desired order. The reader should verify that this constraint is just

$$\bar{\chi}(\mathbf{q}) = \sum_j e^{-i\mathbf{q} \cdot \mathbf{R}_{vj}} = \sum_j \exp(-i\mathbf{q} \cdot (\mathbf{r}_j + \frac{l^2}{2} \hat{\mathbf{z}} \times \mathbf{p}_j)) = 0 \quad (165)$$

expanded to order  $ql$ .<sup>32</sup> To this order the electron density is (dropping  $\mathbf{a}$  terms which do not matter at low energies)

$$\rho^e(\mathbf{q}) = \sum_j e^{-i\mathbf{q} \cdot \mathbf{R}_{ej}} = \sum_j \exp(-i\mathbf{q} \cdot (\mathbf{r}_j - \frac{l^2}{2} \hat{\mathbf{z}} \times \mathbf{p}_j)) \quad (166)$$

$$\simeq \sum_j \exp(-i\mathbf{q} \cdot \mathbf{r}_j) (1 - \frac{il^2}{2} \mathbf{q} \times \mathbf{p}_j) \quad (167)$$

$$= \rho(\mathbf{q}) - \frac{il^2}{2} \mathbf{q} \times \mathbf{g}(\mathbf{q}). \quad (168)$$

Recall the constraint generates gauge transformations. To zeroth order the constraint operator is just  $\sum_j e^{-i\mathbf{q} \cdot \mathbf{r}_j} = 0$  and its action is

$$\mathbf{r}_j \rightarrow \mathbf{r}_j \quad \mathbf{p}_j \rightarrow \mathbf{p}_j + \mathbf{q} e^{-i\mathbf{q} \cdot \mathbf{r}_j}. \quad (169)$$

Thus respecting the constraint ensures  $K$ -invariance.

Stern *et al* (1999) now perform an RPA calculation of the electronic density-density correlation. RPA works because at a given  $\mathbf{q}$ , the gauge field (of that  $\mathbf{q}$ ) enters only in the wiggles connecting irreducible bubbles in Figure (3) and nowhere inside these bubbles: every internal exchange brings with it a sum over  $q$  which introduces a small parameter  $Q$ . Since in the limit  $Q \rightarrow 0$ , RPA diagrams are all we have, RPA respects the symmetry of  $H_{aug}$ . We will now argue that if the constraint is respected, compressibility follows. For the electron density operator we have many choices starting with Eqn. (168) and using the constraint Eqn. (164). In particular we can write it as

$$\rho^e = 2\rho \quad (170)$$

---

<sup>32</sup> Since the (CS) constraint commutes with  $H$  prior to the action of  $U$ , it does so after the action of  $U$ , but of course only to leading order in  $ql$  since  $U$  was not implemented exactly.



or as

$$\rho^e = -2 \frac{il^2}{2} \mathbf{q} \times \mathbf{g}(\mathbf{q}). \quad (171)$$

Written the first way, a nonzero compressibility is not surprising since there are no powers of  $q$  in the operator. In the second way, we see the possible paradox, since there is a  $q$  up front in each of the two factors of  $\rho^e$ . It is here that the overdamped mode appears in the transverse sector, couples to  $\mathbf{g}(\mathbf{q})$  and saves the day with a factor  $q^2 v(q)$  in its static propagator, yielding results that coincide with HLR.

The details of this formidable calculation are not shown here in the interest of brevity. Suffice it to say that five different operators ( $\rho$ , and the two components each of  $\mathbf{g}$  and  $\mathbf{C}$ , where  $\mathbf{C}$  is another vector operator) get coupled and  $K^{-1} = K_0^{-1} + U$  is a relation among  $5 \times 5$  matrices. It is remarkable that in the end all the physics of HLR, including the overdamped mode, finite compressibility and mass divergences at the Fermi surface, all emerge from the more physical quasiparticles, though after a lot of work.

However this is not quite the end since we have only discussed the  $Q \rightarrow 0$  limit, while the actual theory has no such limit on  $Q$ . Stern *et al* (1999) finesse this question with a two-step argument: (i) First they show that  $K$ -invariance guarantees that the Landau parameter  $F_1$  has to be  $-1$ . (ii) Then they show that this condition generally produces very soft modes which restore compressibility to the dipole gas. Let us elaborate a little. Suppose we want define an effective low-energy theory for the composite fermions. How do we ensure that it has  $K$ -invariance? We cannot possibly find all the higher order correction in the  $Q$  expansion. This is also the typical situation in Landau theory, that although in principle the Landau parameters can be calculated given the microscopic interaction, there is no way to do this reliably in practice. However, there are exceptions where a symmetry is involved. For example, Galilean invariance can be used to relate the bare mass, the physical mass, and the Landau parameter  $F_1$ . *It turns out that here too,  $K$ -invariance can be assured for the actual problem if we choose  $F_1 = -1$ .* The reason is that the energy cost of boosting the Fermi sea is measured by  $(1 + F_1)$ . Choosing  $F_1 = -1$ , makes the boost cost-free, i.e., implies  $K$ -invariance. In other words the dipolar fermion cannot have an arbitrary Fermi liquid interaction: its  $F_1$  must equal  $-1$ . At the level of diagrams, if  $F_1 = -1$  is included as an interaction, the correlation function of two dipolar densities will go as  $q^0$  and not  $q^2$ .

The compressibility of the  $\nu = 1/2$  system was also established by Read (1998) using a conserving approximation. He begins with the problem of bosons at unit filling, a problem first studied by Pasquier and Haldane (1998), though not in a conserving approximation. The bosons can be traded for fermions in zero average field by attaching a flux quantum. The role of fluctuations is the same as in the HLR problem except for the strength of the gauge field-fermion coupling. The hamiltonian is just the electronic interaction written in the CF basis with

$$\mathbf{R}_e = \mathbf{r} - \frac{l^2}{2} \bar{\mathbf{z}} \times \mathbf{p} \quad (172)$$

and the constraint is the density corresponding to

$$\mathbf{R}_v = \mathbf{r} + \frac{l^2}{2} \bar{\mathbf{z}} \times \mathbf{p}. \quad (173)$$

The advantage of this starting point is that the constraint exactly commutes with  $\bar{H}$ . Thus one can look for approximation schemes in which the constraint is respected at the level of Green's functions, i.e., Green's functions with any number of  $\bar{\chi}(\mathbf{q})$ 's in them vanish. Read's (1998) calculation begins with the filled Fermi sea (which does not respect the constraint) and embellishes it with diagrammatic corrections. An infinite sum of ladders (whose legs contain fermions in zero field) leads to the collective mode. The density-density correlator has the appearance of dipolar objects that exchange a transverse gauge field. The transverse propagator (which is just the overdamped mode) introduces a  $v(q)q^2$  in the denominator that offsets the  $q^2$  upstairs. It is found that correlators containing  $\bar{\chi}(\mathbf{q})$ 's vanish and Ward identities are satisfied.

It is instructive to look at Read's (1998) density-density correlator which takes the form:

$$K_{irr}(q, \omega) = \int d^2k d^2k' (e^{il^2 \mathbf{k} \times \mathbf{q}} - 1) M(\mathbf{k}, \mathbf{k}', \mathbf{q}, \omega) (e^{-il^2 \mathbf{k}' \times \mathbf{q}} - 1) \quad (174)$$

where the factors at each end reduce to dipoles at small  $ql$  and  $M = M_0 + M_T$  is a sum of two terms, one from the free Fermi sea and one from the exchange of the transverse collective mode which peaks at  $i\omega \simeq q^3 v(q)$ . At high frequencies,  $M_T$  can be ignored and the dipoles emerge as free objects, while at low frequencies, they are coupled by the overdamped mode and do not behave like classical dipoles. We begin to see how the dipoles appear in the correlators (because we can tune the frequency and wavelength in  $K$  to expose them) but not the wavefunctions, which describe equal-time correlations and thus involve an integral over all frequencies.

Arguments for the compressibility of the dipolar system were also given by D.H. Lee (1998).

The resolution of the compressibility paradox did much to assure the community that various descriptions of the quasiparticle, each tailor-made for a different occasion, were mutually compatible and consistent.

### E. Higher Landau Levels

The advantage of the extended hamiltonian is that it keeps track of the electronic cyclotron coordinate and does not go to the LLL prematurely. This has many benefits. If we want to compute the Hall conductance, we can couple the system to an external potential and find the response. As stated previously, the response involves higher LL's so that the presence of  $\eta_e$  is crucial. The details are shown in Appendix B. We can also study the effects of LL mixing. Instead of using just  $H_{LLL}$  we can try to get an effective theory within the LLL which subsumes the effects of virtual transitions to higher LL's.

To extract the leading correction due to higher LL's we write the Schrödinger equation in schematic form as as

$$\begin{bmatrix} H_{00} & H_{0n'} \\ H_{n'0} & H_{n'n'} \end{bmatrix} \begin{bmatrix} \phi \\ \xi \end{bmatrix} = E \begin{bmatrix} \phi \\ \xi \end{bmatrix} \quad (175)$$

where  $\phi$  is restricted to the space spanned by Fock states composed of just the LLL states and  $\xi$  stands for everything else. Likewise the subscripts 0 and  $n'$  stand for collective labels in the LLL and above the LLL respectively. The exact equation obeyed by  $\phi$  is

$$\left( H_{00} + H_{0n'} \frac{1}{E - H_{n'n'}} H_{n'0} \right) \phi = E \phi \quad (176)$$

which is not an eigenvalue problem since  $E$  appears on both sides. However we may approximate as follows:

$$\frac{1}{E - H_{n'n'}} = -\frac{1}{H_{n'n'}} + \mathcal{O}(v/\omega_0) \quad (177)$$

since the eigenvalue  $E$  we are interested in is of order  $v$  and the eigenvalues of  $H_{n'n'}$  are of order  $\omega_0$ . To the same accuracy in  $\kappa = v/\omega_0$  we can also replace

$$H_{n'n'} \simeq H_{n'n'}^0 = \sum_{\alpha} a_{\alpha}^{\dagger} a_{\alpha} n'_{\alpha} \omega_0 \quad (178)$$

which leads to

$$H_{00}^{eff} = H_{00} - \sum_{n'} H_{0n'} \frac{1}{n' \omega_0} H_{n'0} \quad (179)$$

Once we have the effective theory in the LLL, we can switch to the CF formalism: Introduce  $\mathbf{R}_v$ , exchange  $\mathbf{R}_e$  and  $\mathbf{R}_v$  for  $\mathbf{R}$  and  $\boldsymbol{\eta}$  and proceed as before by a Hartree-Fock calculation. The results are in accord with earlier works that showed that LL-mixing reduces the transport gap, but that finite thickness reduces this effect (Yoshioka, 1986, Price, Platzman, and He, 1993, Melik-Alaverdian and Bonesteel, 1995, Price and Das Sarma, 1996, Melik-Alaverdian, Bonesteel, and Ortiz, 1997). Full details can be found in Murthy and Shankar (2002).

### VI. CORRELATION FUNCTIONS IN THE CONSERVING APPROXIMATION

This section illustrates how one does the conserving calculation within the hamiltonian approach by deriving density-density correlation functions (Murthy, 2001a). This calculation differs from Read (1998) in two ways. First, it is done for nonzero effective field ( $\nu \neq 1/2$ ), so that the CF's are in Landau levels instead of plane wave states. Next, the calculation is done in the operator approach (Anderson, 1958a,b, Rickayzen, 1958) using equations of motion as compared to diagrams (Nambu, 1960). The second difference is only cosmetic, and introduced here to promote harmony with the rest of the paper<sup>33</sup>.

The Hamiltonian and constraint to be solved are

$$H = \frac{1}{2} \sum_{\mathbf{q}} \bar{\rho}(\mathbf{q}) v(q) e^{-(ql)^2/2} \bar{\rho}(-\mathbf{q}) \quad (180)$$

$$\bar{\chi}(\mathbf{q}) \simeq 0. \quad (181)$$

Let us define a time-ordered pseudovortex-electron density-density Green's function  $G_{ve}(q, t)$  as follows (with  $G_{ee}$  and  $G_{vv}$  similarly defined):

---

<sup>33</sup> For an illustration of calculations in the gapped fractions using the diagrammatic approach, see Green (2001).

$$\begin{aligned}
G_{ve}(\mathbf{q}, t - t') &= -i \langle T \bar{\chi}(\mathbf{q}, t) \bar{\rho}(-\mathbf{q}, t') \rangle \\
&= -i \Theta(t - t') \langle \bar{\chi}(\mathbf{q}, t) \bar{\rho}(-\mathbf{q}, t') \rangle \\
&\quad - i \Theta(t' - t) \langle \bar{\rho}(-\mathbf{q}, t') \bar{\chi}(\mathbf{q}, t) \rangle
\end{aligned} \tag{182}$$

which evolves as per

$$\begin{aligned}
-i \frac{\partial}{\partial t} G_{ve}(\mathbf{q}, t - t') &= -\delta(t - t') \langle [\bar{\chi}(\mathbf{q}, t), \bar{\rho}(-\mathbf{q}, t')] \rangle \\
&\quad - i \langle T[H, \bar{\chi}(\mathbf{q}, t)] \bar{\rho}(-\mathbf{q}, t') \rangle
\end{aligned} \tag{183}$$

Since  $\bar{\chi}$  commutes with  $H$ , one immediately sees that  $G_{ve}$  is a constant. If the constraint is set to zero initially, then it remains zero, and all its correlators also remain zero.

The above is true in an exact treatment of the theory. Of course, we can only do approximate calculations. A calculation that respects  $G_{ve} = G_{vv} = 0$  respects the symmetries of the theory at the level of correlators, and is called conserving. Let us see what a natural approximation scheme might be. Consider

$$\begin{aligned}
i \frac{\partial}{\partial t} G_{ee}(\mathbf{q}, t) &= \delta(t) \langle [\bar{\rho}(\mathbf{q}, t), \bar{\rho}(-\mathbf{q}, 0)] \rangle \\
&\quad - i \langle T[H, \bar{\rho}(\mathbf{q}, t)] \bar{\rho}(-\mathbf{q}, 0) \rangle
\end{aligned} \tag{184}$$

Since  $[\bar{\rho}, \bar{\rho}] \simeq \bar{\rho}$ , a Green's function involving three densities will arise. Additional time derivatives will produce higher-order density correlators, leading to a hierarchy of equations for more and more complicated Green's functions. The natural way to truncate this hierarchy is to make a mean-field approximation at some stage that reduces a product of two densities to a single density. One of the simplest of such approximations (Baym and Kadanoff, 1961) reduces  $[H, \bar{\rho}]$ , which is a product of four fermi operators, to a product of only two, by using the averages

$$\begin{aligned}
d_{\alpha_1}^\dagger d_{\alpha_2}^\dagger d_{\beta_2} d_{\beta_1} &\rightarrow \langle d_{\alpha_1}^\dagger d_{\beta_1} \rangle d_{\alpha_2}^\dagger d_{\beta_2} + \langle d_{\alpha_2}^\dagger d_{\beta_2} \rangle d_{\alpha_1}^\dagger d_{\beta_1} \\
&\quad - \langle d_{\alpha_1}^\dagger d_{\beta_2} \rangle d_{\alpha_2}^\dagger d_{\beta_1} - \langle d_{\alpha_2}^\dagger d_{\beta_1} \rangle d_{\alpha_1}^\dagger d_{\beta_2}
\end{aligned} \tag{185}$$

Here  $\langle d_{\alpha}^\dagger d_{\beta} \rangle = \delta_{\alpha\beta} N_F(\alpha)$ , where  $N_F(\alpha)$  is the Fermi occupation of the single-particle state  $\alpha$ .

Using the HF states and their occupations in the above truncation is known as the time-dependent HF (TDHF) approximation<sup>34</sup>. We will use the operator approach to TDHF as expounded by Anderson (1958a,b) and Rickayzen (1958). We will explicitly see below that it is conserving for all principal fractions.

The physical picture underlying our calculation is the following. When a bosonic operator such as  $\bar{\rho}(\mathbf{q})$  or  $\bar{\chi}(\mathbf{q})$  acts on the ground state, it creates a linear combination of particle-hole pairs. In the Landau gauge, each pair is labeled by an index  $\nu = n_1, n_2$  (not to be confused with the filling factor!) that keeps track of the CF-LL indices of the particle and hole, and a total pair momentum  $\mathbf{q}$  which is conserved because the particle and hole have opposite charges and do not bend in the magnetic field. It is clear that in order to calculate time-dependent response functions we have to understand the time-evolution of these pairs. In the exact theory, the Hamiltonian can scatter a pair into two pairs, two pairs into four pairs, etc. This is what leads to the hierarchy of equations. However, the great simplicity of the TDHF approximation is that in this approximation a particle-hole pair scatters only into another particle-hole pair. At a given  $\mathbf{q}$  we thus have a matrix labeled by the indices  $(\nu, \nu')$  of the incident and scattered particle-hole pairs. The magnetoexciton spectrum comes from the eigenvalues of this matrix, which, together with its eigenvectors, will be seen to explicitly determine the Green's function.

Clearly, we must begin with an operator that creates an exciton in a state of definite momentum  $\mathbf{q}$ , starting with states labeled by CF-LL indices  $\nu = (n_1, n_2)$ . The following operator does the job:

$$O_{n_1 n_2}(\mathbf{q}) = \sum_X e^{-iq_x X} d_{n_1, X - \frac{q_y l^* 2}{2}}^\dagger d_{n_2, X + \frac{q_y l^* 2}{2}} \tag{186}$$

---

<sup>34</sup> The TDHF approximation has been the method of choice in computing magnetoexciton dispersions in the IQHE. Here are some of the early references: Chiu and Quinn (1974), Horing and Yildiz (1976), Theis (1980), Bychkov, Iordanskii, and Eliashberg (1981), Bychkov and Rashba (1983), Kallin and Halperin (1984), MacDonald (1985), Hawrylak and Quinn (1985), Marmorkos and Das Sarma (1992), Longo and Kallin (1993).

Why is this so? First note that in this gauge, (Eqn.(45)), the wavefunctions are plane waves in  $y$  with momentum  $k$  and localized in  $x$  at  $X = kl^{*2}$ . Thus  $d_{m_1, X - \frac{q_y l^{*2}}{2}}^\dagger d_{m_2, X + \frac{q_y l^{*2}}{2}}$  which creates a hole at  $X - \frac{q_y l^{*2}}{2}$  and a particle at  $X + \frac{q_y l^{*2}}{2}$ , creates an exciton in a state of momentum  $q_y$  in the  $y$ -direction, centered at  $X$ . Multiplying by  $e^{-iq_x X}$  and summing over  $X$  creates a state of momentum  $q_x$  in the  $x$ -direction.

Now we define

$$G(\nu; \nu'; \mathbf{q}; t - t') = -i \langle T O_\nu(\mathbf{q}, t) O_{\nu'}(-\mathbf{q}, t') \rangle. \quad (187)$$

Taking the time derivative we get

$$\begin{aligned} -i \frac{\partial}{\partial t} G(\nu; \nu'; \mathbf{q}; t) &= -i \langle T [H, O_\nu(\mathbf{q}, t)] O_{\nu'}(-\mathbf{q}, 0) \rangle \\ &\quad - \delta(t) \langle [O_\nu(\mathbf{q}, t), O_{\nu'}(-\mathbf{q}, 0)] \rangle \end{aligned} \quad (188)$$

The last piece is the standard inhomogeneous “source” term. The dynamics is controlled by the commutator with the Hamiltonian, to which we now turn

$$\begin{aligned} [H, O_\nu(\mathbf{q})] &= (\epsilon(n_1) - \epsilon(n_2)) O_\nu(\mathbf{q}) + \\ &\quad (N_F(n_2) - N_F(n_1)) \sum_{\nu''} \left( \frac{v(q)}{2\pi(l^*)^2} e^{-q^2 l^2 / 2} \rho_{n_1'' n_2''}(\mathbf{q}) \times \rho_{n_2 n_1}(-\mathbf{q}) \right. \\ &\quad \left. - \int \frac{d^2 s}{(2\pi)^2} v(s) e^{-s^2 l^2 / 2} \rho_{n_1'' n_1}(\mathbf{s}) \rho_{n_2 n_2''}(-\mathbf{s}) e^{i(l^*)^2 \mathbf{s} \times \mathbf{q}} \right) O_{\nu'}(\mathbf{q}) \end{aligned} \quad (189)$$

where  $\epsilon(\nu)$  is the Fock energy in a state of CF-LL index  $\nu$ , and the TDHF approximation has been made.

Note that the action of commuting with  $H$  on  $O_\nu$  in the TDHF approximation can be represented as the right-multiplication by a matrix  $\mathcal{H}(\nu; \nu''; \mathbf{q})$ . It follows that if we form a linear combination of operators  $O_\Psi = \sum_\nu \Psi(\nu; \mathbf{q}) O_\nu$ , the column vector  $\Psi$  will transform linearly under the action of  $\mathcal{H}$ . Diagonalizing  $\mathcal{H}$  will also enable us to solve for the Green’s function, for the eigenvectors represent linear combinations of particle-hole states that are normal modes of  $\mathcal{H}$ . Assume that one has found the right and left eigenvectors and corresponding eigenvalues  $E_\alpha(q)$ , labeled by  $\alpha$

$$\mathcal{H}(\nu; \nu''; \mathbf{q}) \Psi_\alpha^R(\nu''; \mathbf{q}) = E_\alpha(q) \Psi_\alpha^R(\nu; \mathbf{q}) \quad (190)$$

$$\Psi_\alpha^L(\nu; \mathbf{q}) \mathcal{H}(\nu; \nu''; \mathbf{q}) = E_\alpha(q) \Psi_\alpha^L(\nu''; \mathbf{q}) \quad (191)$$

where sums over repeated indices are implicit.

Assuming the matrix  $\mathcal{H}$  has a complete set of eigenvectors the Green’s function can be written as

$$G(\nu; \nu'; \mathbf{q}; \phi) = \frac{L^2}{2\pi(l^*)^2} \sum_\alpha \Psi_\alpha^R(n_1 n_2; \mathbf{q}) \frac{1}{\phi - E_\alpha} \Psi_\alpha^L(n'_1 n'_2; \mathbf{q}) (N_F(n'_1) - N_F(n'_2)) \quad (192)$$

where the factor  $(N_F(n'_1) - N_F(n'_2))$  comes from the source term (the last term of Eq. (188)).

An important property of  $\mathcal{H}$  from the point of view of the conserving approximation is that it always has one left eigenvector with zero eigenvalue for every  $\mathbf{q}$ , namely

$$\Psi_0^L(n_1 n_2; \mathbf{q}) = \tilde{\chi}_{n_1 n_2}(\mathbf{q}) \quad (193)$$

where

$$\tilde{\chi}_{n_1 n_2}(\mathbf{q}) = \langle n_1 | e^{-i\mathbf{q} \cdot (\boldsymbol{\eta}/c)} | n_2 \rangle. \quad (194)$$

The existence of this eigenvector with zero eigenvalue (for any  $v(q)$ ) is shown explicitly in Murthy (2001a). This zero eigenvalue is related to the constraint and is one of the conditions for the TDHF approximation to be conserving. To see this let us go back to the exact theory. There the Hamiltonian commutes with  $\bar{\chi}(\mathbf{q})$ . This means that  $\bar{\chi}(\mathbf{q})$  acting on the ground state should produce a *zero energy* state at every  $\mathbf{q}$ . In the TDHF approximation the action of the Hamiltonian on this state is approximated by the action of the matrix  $\mathcal{H}$  on the left vector  $\Psi_0^L(n_1 n_2; \mathbf{q})$ , corresponding to the operator  $O_{\bar{\chi}(\mathbf{q})} = \sum \tilde{\chi}_{n_1 n_2}(\mathbf{q}) O_{n_1 n_2}(\mathbf{q})$ . The fact that  $\mathcal{H}$  admits  $\Psi_0^L$  as a left eigenvector with *zero eigenvalue* means that the zero energy state which had to be present in the exact theory is also present in the TDHF approximation.

To put it in slightly different words, the TDHF approximation represents a truncation of the Hilbert space of neutral excitations to states having only a single particle-hole pair above the ground state. *A priori* this truncation need not have respected the constraint, but it does.

The above condition is necessary for TDHF to be conserving, but not sufficient. The other condition is that the physical sector (excitations created by  $\bar{\rho}(\mathbf{q})$ ) does not couple to the zero energy sector of  $\mathcal{H}$ . This can also be verified in a straightforward way (Murthy 2001a).

### A. Small- $q$ structure factor

We will now put the conserving approximation to another test. Recall that the density-density correlator has two parts; one coming from the cyclotron pole, with a residue of  $q^2$ , which is unrenormalized as per Kohn's theorem and saturates the sum rule, and another coming from the dynamics in the LLL. Hence the LLL-projected structure factor  $\bar{S}(q)$  has to vanish faster than  $q^2$ , and if it is analytic in  $q^2$ , it has to go like  $q^4$ . We will see if this is true. Luckily, to obtain the leading behavior of  $\bar{S}(q)$  in the TDHF approximation one needs to keep only a finite-dimensional submatrix of the infinite TDHF matrix.

The results are as follows. For  $\nu = \frac{1}{3}$  we have, upon diagonalizing a  $4 \times 4$  matrix,

$$\bar{S}(q) = \frac{(ql)^4}{8} + \dots \quad (195)$$

We can extend this result to all the Laughlin fractions  $\frac{1}{2s+1}$  to obtain

$$\bar{S}(q) = \frac{1}{8}(ql)^4 + \dots \quad (196)$$

The coefficient  $\frac{1}{8}$ , independent of  $s$ , differs from the result  $\frac{s}{4}$  obtained from Laughlin's wavefunction (Girvin, MacDonald, and Platzman, 1986). However no general principle requires that we regain this coefficient.

One can carry out a very similar, but much more tedious, calculation for all the principal fractions (Murthy, 2001a). The result, upon diagonalizing a  $6 \times 6$  matrix, is

$$\bar{S}(q) = \frac{(ql)^4}{2} \frac{p^4 - 3p^3 + \frac{5}{4}p^2 + 3p + \frac{7}{4}}{p^2 - 1} \quad (197)$$

This expression, with its divergence as  $p \rightarrow \infty$  or  $\nu \rightarrow \frac{1}{2}$  is consistent with the result (Read, 1998) that for a problem equivalent to the  $\nu = \frac{1}{2}$  problem,  $\bar{S}(q) \simeq q^3 \log(q)$ . As  $p \rightarrow \infty$ , the radius of convergence of the power series expansion of  $\bar{S}(q)$  must go to zero (else the structure factor would diverge for a range of  $q$  according to Eqn. (197)), and the  $p \rightarrow \infty$  limit does not commute with the  $q \rightarrow 0$  limit. The formula also does not hold for  $p = 1$ , the Laughlin fractions, for which we must use the results quoted earlier.

Surprisingly, the results are independent of the form of the potential, which does enter the intermediate stages of the calculation.

### B. Magnetoexciton dispersions for $\frac{1}{3}$ and $\frac{2}{5}$

We saw that the small- $q$  behavior of  $\bar{S}(q)$  can be satisfactorily addressed using the TDHF approximation. It turns out that this approximation also works well for computing the dispersion of magnetoexcitons, by which we mean the lowest energy physical eigenstate of  $\mathcal{H}$  at each  $\mathbf{q}$ .

As shown in the previous section, for very small  $q$ , the naive magnetoexcitons (bare particle-hole states created by  $O_\nu$ ) do not mix with others and become the true eigenstates of  $\mathcal{H}$ . As  $q$  increases they become increasingly coupled, and both level repulsion (between positive energy states) and level attraction (between positive and negative energy states) manifest themselves, giving rise to complicated magnetoexciton dispersions. The matrix  $\mathcal{H}$  is infinite-dimensional, but in any numerical calculation, only a finite matrix can be diagonalized. We saw that the lowest nontrivial result for  $\bar{S}(q)$  could be obtained by keeping at most a  $6 \times 6$  matrix. As  $q$  increases more and more CF-LLs have to be kept to obtain accurate results. The accuracy of the truncation was checked by two different methods. Firstly, the number of CF-LLs kept was increased until the energy of the magnetoexciton was stable. Secondly, we knew that at every  $q$  there should be two zero eigenvalues corresponding to the unphysical sector. The number of CF-LLs kept in the calculation was increased until these null eigenvalues were at least four orders of magnitude smaller than the smallest physical eigenvalue.

Figures 4 and 5 are for the potential  $v(q) = \frac{2\pi e^2}{\epsilon q} e^{-q^2 \beta^2 / 2}$  for various values of  $\beta$ . It is worth noting that though the depths of the minima are  $\beta$ -dependent, and different from what are found in numerical diagonalizations or from

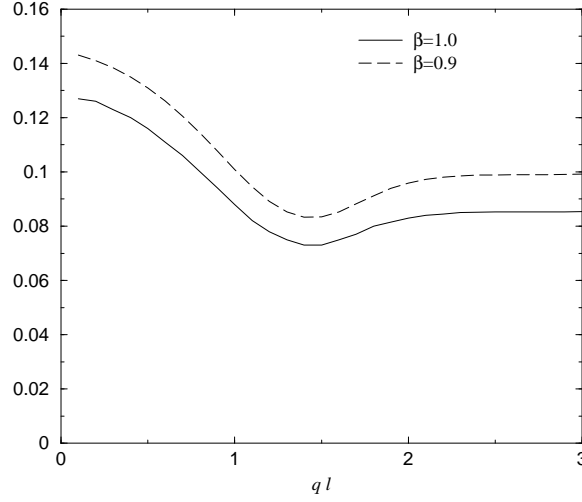


FIG. 4 The lowest-energy spin-polarized magnetoexciton for  $\nu = 1/3$  for two values of  $\beta$ . Note the "magnetoroton" at  $ql = 1.4$ .

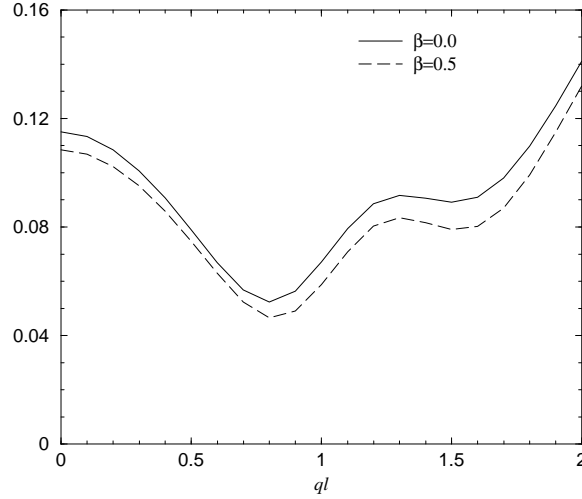


FIG. 5 The lowest-energy spin-polarized magnetoexciton for  $\nu = 2/5$  for two values  $\beta$ . Note the minima at  $ql = 0.8$  and  $ql = 1.5$ .

CF-wavefunctions, the positions are correct<sup>35</sup>. This indicates that the TDHF approximation to the EHT does capture the important physics of the electronic problem even at fairly large  $q$ .

In the case of  $\frac{1}{3}$ , the magnetoexciton energy has stabilized at large  $ql$  enabling us to read off the *activation gap*,  $\Delta_a$ , defined as the minimum energy needed to produce a widely separated particle-hole pair. However, the situation deteriorates rapidly as  $p$  increases. It becomes prohibitively hard, even at  $p = 2$ , to get the large  $ql$  limit of the magnetoexciton spectrum, as is evident from Figure 5. This is because the decoupling of the naive magnetoexcitons is only asymptotic for large  $q$ . However, as  $p$  increases, the *shortcut* gives very good answers (Figures 6 and 7) with hardly any additional work. We now turn to it.

<sup>35</sup> Haldane and Rezayi (1985), Su and Wu (1987), Morf and Halperin (1987), d'Ambrumenil and Morf (1989), He, Simon, and Halperin (1994), He and Platzman (1996), Kamilla, Wu, and Jain (1996a,b), Jain and Kamilla (1998).

## VII. GAPS

Having developed the EHT at length and having established several qualitative results and matters of principle, we now turn to mainly quantitative issues. As mentioned at the outset, one can compute just about anything within this approach, some more accurately than others. In the next few sections we will discuss a variety of such quantities. Rather than run through an endless list we will focus on a few that give the flavor of the method and expose its strengths and shortcomings, and point out references containing more details or examples. An invaluable benchmark will be provided by results from trial wavefunctions and exact diagonalization. Comparison with experiment will be done with the clear understanding that no systematic attempt is made to incorporate disorder.

In comparing to the results of exact diagonalization and trial wavefunctions one must bear in mind their limitations. Exact diagonalization (ignoring machine errors) suffers from the fact that the systems are necessarily small. In comparing these results to ours (valid for infinite systems) we must examine the approach to the thermodynamic limit and the scaling of gaps. We refer the reader to Morf, d'Ambrumenil, and Das Sarma (2002) for the most recent study of this kind.

While trial wavefunctions can be written down for any number of particles, the evaluation of gaps requires once again that the number of particles be small (though not as small as in exact diagonalization). The details may be found in Park, Meskini and Jain (1999) to whom we compare our numbers. It must also be borne in mind that the correct wavefunctions may not be of the form being tried, in which case even a flawless evaluation of gaps is irrelevant. So far there is good reason to believe that for  $1/3 < \nu \leq 1/2$  the Jain-like functions are good. At very low densities the correct state is believed to be a Wigner crystal and when higher LL's are involved, more complicated functions like Pfaffians (the Moore-Read, 1991, state) need to be explored. With these caveats in mind we proceed with the comparison of our theory with numerics.

*In the remaining sections, we will use the preferred density and pay no further regard to constraints.* As stated earlier, this is the most efficient way to work on problems that do not depend crucially on the deep infrared region ( $\omega \simeq q^3 v(q)$ ). In the cases discussed this is ensured by either a gap, a nonzero temperature, or both.

Let us begin with the activation or *transport gap* in a fully polarized sample, defined as the minimum energy needed to produce a widely separated particle hole pair:

$$\Delta = \langle \mathbf{p} + PH | H | \mathbf{p} + PH \rangle - \langle \mathbf{p} | H | \mathbf{p} \rangle \quad (198)$$

where  $|\mathbf{p}\rangle$  stands for the Hartree-Fock ground state with  $p$ -filled LL's and  $PH$  stands for a widely separated particle-hole pair. We will use a boldface symbol such as  $\mathbf{p}$  to label a Slater determinant with  $p$  occupied Landau Levels. Non-boldface symbols will label single-particle states. Note also that the highest occupied CF-LL index  $n$  for the state labeled by  $p$  is  $n = p - 1$  since the CF-LLL has index  $n = 0$ . As shown in Appendix C, the particle-hole excitations of  $|\mathbf{p}\rangle$  are HF states of our  $H$ .

The expression for the gap written above is formally the same in the wavefunction based approach of Park, Meskini and Jain (1999) to whose results we will compare ours. However the notation hides a big difference. They work in the *electronic basis* where  $\rho(\mathbf{q}) = \sum_j \exp(-i\mathbf{q} \cdot \mathbf{r}_j)$ , and the states are the simple wavefunctions  $\chi_p$ , multiplied by the Jastrow factor and then projected to the LLL. (Projection leads to a very complicated expression for the wavefunctions.) In the present approach we have tried to incorporate these effects by going in the reverse direction, from electrons to CF's, and obtaining complicated expressions for the charge and other operators, but with simple expressions for the wavefunctions. While these operator expressions are unusual in form, they are simple to evaluate within the HF calculation.

In all our calculations we shall use the Zhang-Das Sarma potential (Zhang and Das Sarma, 1986)

$$v_{ZDS} = \frac{2\pi e^2 e^{-q\lambda}}{q} \quad (199)$$

which is a crude model for the electron-electron interaction in a sample of finite thickness. We simply take it to be a one parameter family of potentials. In terms of the Haldane pseudopotentials  $V_m$  (which gives the interaction in a state of relative angular momentum  $m$ , see Haldane, 1987) we know that just one (typically  $V_1$ ) dominates. We can think of  $\lambda$  as controlling the operative pseudopotential.

Rather than work with a widely separated particle-hole pair, we find the energy in a state with just the particle and add to it the energy of a state with just the hole and subtract double the ground state energy. Relegating the details to Appendix E, we present the central idea.

We begin with the second quantized expression for the preferred charge operator  $\bar{\rho}^p(\mathbf{q})$ :

$$\bar{\rho}^p(\mathbf{q}) = \sum_{m_2 n_2; m_1 n_1} d_{m_2 n_2}^\dagger d_{m_1 n_1} \rho_{m_2 n_2; m_1 n_1}(\mathbf{q}) \quad (200)$$

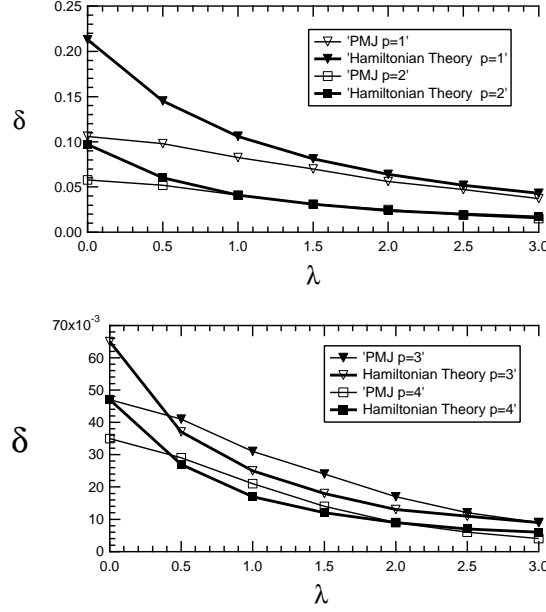


FIG. 6 Comparison of dimensionless activation gaps  $\delta_a$  to the work of Park, Meskini and Jain (1999) for the fractions  $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}$  ( $p = 1, 2, 3, 4$ , and  $s=1$ ) as a function of  $\lambda$ , the thickness parameter in the ZDS potential.

where  $d_{mn}^\dagger$  creates a particle in the state  $|m n\rangle$  where  $m$  is the angular momentum and  $n$  is the LL index of CF in the weakened field  $A^* = A/(2ps + 1)$  with a magnetic length

$$l^* = l\sqrt{2ps + 1}. \quad (201)$$

The key ingredient in the HF calculation is the matrix element  $\rho_{m_2 n_2; m_1 n_1}$  which factorizes (as shown in Appendix A):

$$\rho_{m_2 n_2; m_1 n_1} = \rho_{m_2 m_1}^m \otimes \rho_{n_2 n_1}^n \quad (202)$$

The gaps depend only on  $\rho_{n_2 n_1}^n$ , the superscript on which will be generally dropped. Often we will use the *dimensionless activation gap*  $\delta_a$  defined by

$$\Delta_a = \frac{e^2}{\varepsilon l} \delta_a. \quad (203)$$

Figure 6 shows the gaps computed for  $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}$  and  $\frac{4}{9}$  for the ZDS potential and compared to the work of Park, Meskini, and Jain (1999) in the region  $0 \leq \lambda \leq 3$ . The following features are noteworthy.

- At  $\lambda = 0$ , the Coulomb case, the gaps are finite in contrast to the small- $q$  theory (Murthy and Shankar, 1999). This is due to the gaussian factor  $e^{-q^2 l^2/2}$  in  $H(\bar{\rho})$  (Eqn. (61)) which was absent in the *small-q* theory. The slope of the graphs in the present theory is nonzero at this point. It is readily verified that  $d\Delta/d\lambda$  at  $\lambda = 0$  is the gap due to a delta-function potential, and should vanish for spinless fermions. The description of CF in terms of  $\bar{\rho}^p$  does not give good answers for potentials as short ranged as the delta function, as anticipated earlier. Indeed even the Coulomb interaction is too singular, and the theory begins to work well only beyond  $\lambda \simeq 1$ .
- Beyond  $\lambda \simeq 1$  the agreement is quite fair in general and best for  $\frac{2}{5}$ . However we cannot go to too large a  $\lambda$  since in this range the system may not be an FQHE state.
- The gaps which do not vanish for any fraction and any finite  $\lambda$ , exceed the Park, Meskini, and Jain (1999) values for  $\frac{1}{3}$  and  $\frac{2}{5}$  and lie below them for  $\frac{3}{7}$  and  $\frac{4}{9}$ . This result is at odds with the general belief that HF always overestimates the gaps by neglecting fluctuations.



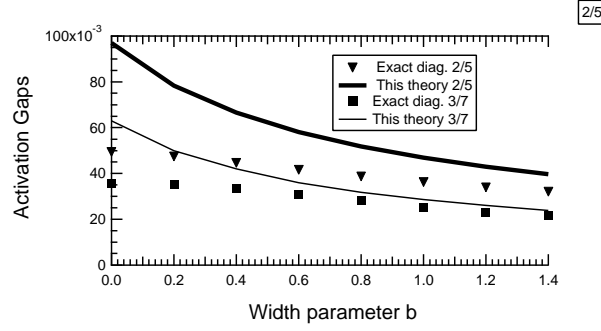


FIG. 7 Comparison of hamiltonian theory to the exact diagonalization results of Morf *et al* for  $p = 2$  and  $3$ .

The situation is different when we compare with the exact diagonalization results of Morf, d'Ambrumenil, and Das Sarma (2002) who used a potential

$$v(q) = \frac{2\pi e^2}{q} e^{(qlb)^2} \text{Erfc}(qlb) \quad (204)$$

where  $b$  is the analog of  $\lambda$ <sup>36</sup>. Our numbers are compared in Figure 7. The calculated gaps always lie above the exact diagonalization results for the two fractions shown (as well as for the  $\frac{1}{3}$  case, not shown).

The general disagreement with our theory is worse for this potential than for the ZDS case because at large  $q$  this potential goes as  $1/q$  while the ZDS potential falls exponentially.

#### A. Activation gaps

We have computed gaps for many other fractions, including for  $s = 2$ , when four vortices are attached to form CF's. Rather than show more plots, we will now analyze the theory in terms of the *activation mass*  $m_a$  defined by

$$\Delta_a = \frac{eB^*}{m_a} = \frac{eB}{(2ps+1)m_a}. \quad (205)$$

Comparison to Eqn. (203) shows that

$$\frac{1}{m_a} = \frac{e^2 l}{\varepsilon} \delta_a (2ps+1) \equiv \frac{e^2 l}{\varepsilon} C_a. \quad (206)$$

Thus

$$C_a = \delta_a (2ps+1). \quad (207)$$

Based on the exact diagonalization results of d'Ambrumenil and Morf (1989) Halperin, Lee, and Read (1993) pointed out that  $C_a$  approaches a limit as we approach  $\nu = \frac{1}{2}$  or  $p \rightarrow \infty$ . The Halperin, Lee, Read theory expects  $C_a$  to be modified by logarithms. We too expect these logarithms once we approach  $\nu = \frac{1}{2}$  and are forced to include the overdamped mode. However, we find as did Halperin, Lee, and Read (1993), a good fit to the calculated gaps without invoking the logarithms, which are operative in a very tiny region near  $\nu = \frac{1}{2}$  (Morf, d'Ambrumenil, and Das Sarma, 2002). The only difference here is that  $C_a$  approaches a limit that depends on  $\lambda$ , a parameter they set equal to zero (Coulomb case).

We next turn to Pan *et al* (2000) whose experiments detected that the *normalized mass* defined by

$$m_a^{nor} = \frac{m_a}{m_e \sqrt{B(T)}} \quad (208)$$

where  $m_e$  is the electron mass and  $B(T)$  is the field in Tesla, is nearly the same for  $s = 1$  and  $s = 2$ , i.e., two and four flux tubes. The theory predicts that  $m^{nor}$  are comparable for  $s = 1$  and  $s = 2$ , but makes it clear that no fundamental

<sup>36</sup> This potential is not very different in form from the ZDS potential except at very short distances.

TABLE II Activation gaps as a function of  $\lambda$  for  $1 \leq \lambda \leq 2$  according to the hamiltonian theory. Note the convergence of  $C_a^{(2)}$  as  $p \rightarrow \infty$ . The superscript (2) refers the number of flux tubes attached.

| p | $\Delta_a^{(2)}/k_B = 50\sqrt{B(T)}\delta_a^{(2)}$ | $\delta_a^{(2)}$ | $C_a^{(2)}$   |
|---|--|------------------|---------------|
| 1 | $5.31\sqrt{B(T)}/\lambda$                          | $.106/\lambda$   | $.32/\lambda$ |
| 2 | $2.08\sqrt{B(T)}/\lambda$                          | $.042/\lambda$   | $.21/\lambda$ |
| 3 | $1.23\sqrt{B(T)}/\lambda$                          | $.025/\lambda$   | $.17/\lambda$ |
| 4 | $0.87\sqrt{B(T)}/\lambda$                          | $.017/\lambda$   | $.16/\lambda$ |

TABLE III Approximate numbers used in this paper, with  $k_B$  the Boltzmann's constant,  $m_e$  the electron mass, and  $B(T)$  the field in Tesla.

|   |  |   |
|---|--|---|
| $\frac{eB}{m_e k_B} = 1.34B(T)^\circ K$ | $\frac{e^2}{\epsilon l k_B} = 50\sqrt{B(T)^\circ K}$ | $\frac{\epsilon}{e^2 l} = .026 m_e \sqrt{B(T)}$ |
|---|--|---|

significance can be attached to this result since it depends on  $\lambda$ , or more generally, the potential. Further details may be found in Shankar (2001).

Consider next the experiments of Du *et al* (1993), who have extensive data on activation gaps. We will limit ourselves to  $\nu \leq \frac{1}{2}$ , to which states with  $1 \geq \nu \geq \frac{1}{2}$  are related by particle-hole symmetry if full polarization is assumed. Given that the experiments, unlike Park, Meskini, and Jain (1999), have an unknown contribution from LL mixing and impurities, it is not clear how to apply the theory. There is no *ab initio* calculation that includes these effects. (There is however reliable evidence that LL mixing is a very small effect at the values of  $\lambda$  under consideration. Recall also our results from Section V.)

We will compute gaps using the ZDS potential with  $\lambda$  as a free parameter, and ask what  $\lambda$  fits the data, just to get a feel for its size. The results are summarized in Table IV.

Comparing the above values of  $\lambda$  extracted from data to the Local Density Approximation (LDA) (Price and Das Sarma, 1996, and references therein) and exact diagonalization calculations (Park, Meskini, and Jain, 1999, Morf, d'Ambrumenil, and Das Sarma, 2002), which suggest  $\lambda \simeq 1$ , we see that disorder has a substantial effect on activation gaps.

It is possible to compute the charge density in a state with a widely separated particle-hole pair in some gapped fractions. The details, and a comparison to the unpublished work of Park and Jain may be found in Shankar (2001).

## B. Other potentials

Figure 8 shows a comparison to the Park, Meskini, and Jain (1999) results for a gaussian potential

$$v(q) = 2\pi e^2 l e^{-q^2 l^2/2} \quad (209)$$

Note that except for  $\nu = \frac{1}{3}$  the agreement is exceptional. This is the kind of ultraviolet-soft potential for which the present theory works best, though unfortunately we do not presently know of any system where it is operative. At present, its study serves to instruct us on the domain of validity of our approach. Not surprisingly, work on the potential,

$$v(r) = \frac{e^{-\kappa r}}{r} \quad (210)$$

shows an agreement that is worse than for the Coulomb case since this potential is just as bad as  $r \rightarrow 0$  and does not give the large  $r$  values a chance. Likewise  $1/r^2$  fares worse than  $1/r$ .

Note that if  $\lambda \rightarrow \infty$  we may be in trouble since the potential may not lead to FQHE states.

TABLE IV Comparison of activation masses to Du *et al*, sample A, which has a density  $n = 1.12 \cdot 10^{11} cm^{-2}$ . The last column gives the best fit to  $\lambda$ .

| $\nu$         | $B(T)$ | $\Delta_a^{exp} (^{\circ}K)$ | $\Delta_a^{theo} (^{\circ}K)$ | $\lambda$ |
|---------------|--------|------------------------------|-------------------------------|-----------|
| $\frac{1}{3}$ | 13.9   | 8.2                          | $5.3\sqrt{B(T)}/\lambda$      | 2.4       |
| $\frac{2}{5}$ | 11.6   | 3                            | $2.08\sqrt{B(T)}/\lambda$     | 2.4       |
| $\frac{3}{7}$ | 10.8   | 2                            | $1.23\sqrt{B(T)}/\lambda$     | 2.0       |

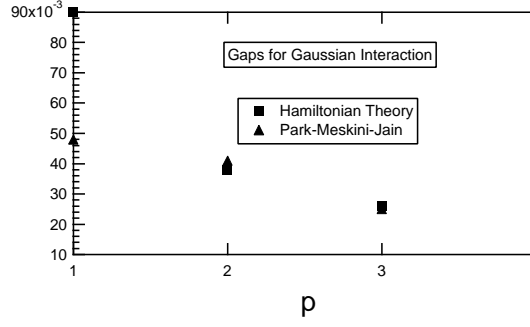


FIG. 8 Comparison of gaps for the gaussian potential. For  $p > 1$  it is hard to separate the results from the Hamiltonian theory and the wavefunction results of Park *et al.*

*In summary, it appears as if our approach works well in any problem which is not sensitive to distances smaller than about a magnetic length.* Since the extended formalism is mathematically equivalent to the original electronic problem in the LLL; it is the twin approximations – the use of  $\bar{\rho}^p$  to deal with constraints and Hartee-Fock – that are responsible for deviations from our benchmarks.

While our approach can reproduce the numbers of the wavefunction and exact diagonalization approaches to within about 10%, (if  $\lambda > 1$ ), with a lot less work (a few seconds on a PC), without these benchmarks, we would have known neither its range of validity nor its degree of accuracy. Without such feedback, we would not have applied the method with the such confidence to other phenomena not treated by wavefunctions or exact diagonalization such as relaxation rates and polarization at  $T > 0$ .

### VIII. MAGNETIC TRANSITIONS AT $T = 0$ .

Now we turn to the behavior of the spin of the system, which so far was assumed to be frozen along the applied field. The reader might wonder why we bother, since the magnetic fields are of order at least a few Tesla. The reason (originally pointed out by Halperin (1983)) is that the Zeeman energy  $E_Z = g^* (\frac{e}{2m}) B$  is the smallest energy scale in the problem. This is because of a conspiracy of two factors: First, due to band structure effects, the band mass of the electron in *GaAs* is  $0.068m_e$ , with  $m_e$  the electron's mass in vacuum. This makes the cyclotron frequency about 14 times what it would have been in empty space. Next, due to spin-orbit coupling, the effective  $g$ -factor of the electron in *GaAs* is  $g = 0.44$  instead of the empty space value of 2. Thus, the Zeeman energy is about 64 times smaller than the cyclotron energy  $\omega_0$ . Typically the interaction energy is of the same order as  $\omega_0$ , leading to  $E_Z$  being the smallest energy. Since the original realization by Halperin (1983), spin transitions have been seen in experiment<sup>37</sup>, and been explored by exact diagonalization<sup>38</sup>. We will compare our results with the recent work of Park and Jain (1998, 1999) in the CF-wavefunction approach.

The coupling of electron spin to the applied field is given by the Zeeman term

$$H_Z = -g \left( \frac{e}{2m_e} \right) \frac{S}{2} B \quad (211)$$

where  $g = .44$ ,  $m_e$  is the electron mass in free space,  $S$  is given by

$$S = n P \quad (212)$$

where  $n$  is the density and  $P$  is the polarization, to which each electron contributes  $\pm 1$ .

Since the uniform external field couples to the  $q = 0$  component of the spin density which is unaffected by the canonical transformations,  $H_Z$  will have the same form in the final CF representation.

#### A. Magnetic transitions in gapped fractions

When  $H_Z$  is large, we expect the system to be fully polarized ( $P = 1$ ). As we lower  $H_Z$ , we expect  $P$  to drop. If translationally invariant CF states are considered for the gapped fractions, there is a discrete set of allowed values of

<sup>37</sup> Clark, Haynes, Suckling, Mallett, Wright, Harris, and Foxon (1989), Eisenstein, Störmer, Pfeiffer, and West (1989), Furneaux, Syphers, and Swanson (1989), Buckthought, Boulet, Sachrajda, Wasilewski, Zawadski, and Guillon (1991), Du, Yeh, Störmer, Tsui, Pfeiffer, and West, (1995,1997).

<sup>38</sup> Chakraborty and Zhang (1984a,b), Rasolt, Perrot, and MacDonald (1985), Chakraborty, Pietiläinen, and Zhang (1986).

$P$ . At  $\nu = p/(2ps + 1)$ , these correspond to states of the form  $|\mathbf{p} - \mathbf{r}, \mathbf{r}\rangle$  in which  $p - r$  CF-LL's are occupied by up spins and  $r$  CF-LL's by down spins.<sup>39</sup> Thus the allowed values of polarization are given by

$$P = \frac{p - 2r}{p}. \quad (213)$$

For example, when  $p = 4$ , the allowed values are  $P = 1, .5$ , and  $0$  corresponding to  $|4, 0\rangle$ ,  $|3, 1\rangle$  and  $|2, 2\rangle$ .

Our goal is to calculate the critical fields at which the system will jump from one value of  $r$  to the next as  $H_Z$  is varied. Let

$$E(p - r, r) = \langle \mathbf{p} - \mathbf{r}, \mathbf{r} | H | \mathbf{p} - \mathbf{r}, \mathbf{r} \rangle \quad (214)$$

where  $H$  does not contain the energy due to  $H_Z$ . This will be the case for the single-particle and ground state energies, with one exception which will be clearly pointed out. Since  $H_Z$  is diagonal in the HF states which have definite spin, its effects can be trivially incorporated.

The HF calculation of  $E(p - r, r)$  is detailed in the Appendix F.

The critical field  $B^c$  for the transition from  $r$  to  $r + 1$  is given by :

$$E(p - r, r) - E(p - r - 1, r + 1) = g \frac{eB^c}{2m_e} \frac{n}{p} \quad (215)$$

where the right hand side denotes the Zeeman cost of flipping the  $n/p$  spins in the LL that switched its spin. This discussion assumes that  $B$  is perpendicular to the sample. If there is a tilt  $\theta$ , we write

$$E(p - r, r) - E(p - r - 1, r + 1) = g \frac{eB_{\perp}^c}{2m_e \cos \theta} \frac{n}{p} \quad (216)$$

When these energy differences are calculated, the same remarkable regularity first noted by Park and Jain (1998, 1999) emerges: they can be fit by a theory of free fermions of mass  $m_p$  the *polarization mass*, that occupy LL's with a gap  $\Delta_p = eB^*/m_p$ . By this we mean the following. In a free theory of gap  $\Delta_p$ , we would have

$$E(p - r, r) - E(p - r - 1, r + 1) = \frac{n(p - 2r - 1)}{p} \Delta_p \quad (217)$$

since  $(n/p)$  spin-up fermions of energy  $(p - r - 1 + \frac{1}{2})\Delta_p$  drop to the spin-down level with energy  $(r + \frac{1}{2})\Delta_p$ . Suppose we evaluate the left-hand-side of Eqn. (217) in the HF approximation and *define*

$$\Delta_p(r)^{def} = \frac{p}{n} \frac{E(p - r, r) - E(p - r - 1, r + 1)}{p - 2r - 1}. \quad (218)$$

Given that  $H$  is not free, there is no reason why  $\Delta_p(r)^{def}$  should be  $r$ -independent. But it is very nearly so. For example at  $p = 6, \lambda = 1$ ,

$$\Delta_p(0, 1, 2)^{def} = \frac{e^2}{\varepsilon l} (0.00660, 0.00649, 0.00641) \quad (219)$$

which describe the transitions  $|6, 0\rangle \rightarrow |5, 1\rangle$ ,  $|5, 1\rangle \rightarrow |4, 2\rangle$ , and  $|4, 2\rangle \rightarrow |3, 3\rangle$ . This  $r$ -independence of the gaps was true for every fraction and every value of  $\lambda$  we looked at. We will soon demystify this apparent free-field behavior, which has a counterpart in the gapless case as well.

The transition  $|\mathbf{p} - \mathbf{r}, \mathbf{r}\rangle \rightarrow |\mathbf{p} - \mathbf{r} - \mathbf{1}, \mathbf{r} + \mathbf{1}\rangle$  occurs when the spin-flip energy equals the energy difference of the two competing ground states:

$$g \frac{e}{2m_e} \frac{B_{\perp}^c}{\cos \theta} = (p - 2r - 1) \Delta_p. \quad (220)$$

---

<sup>39</sup> In Jain's approach, the actual wavefunction will be such a state times the Jastrow factor, followed by projection to the LLL. In the present approach,  $|\mathbf{p} - \mathbf{r}, \mathbf{r}\rangle$  is literally the state, but the operators for charge and spin are obtained by transformations to the CF basis. For the interested reader we mention that these transformations are spin-independent.

## B. Magnetic properties of gapless fractions

Let us now turn to the gapless fractions  $\frac{1}{2}$  and  $\frac{1}{4}$ . The discrete labels  $p - r$  and  $r$  of the HF states that count the spin-up and down LL's are now replaced by continuous variables  $k_{\pm F}$  which label the Fermi momenta of the spin-up and down seas. These momenta are such that the total number of particles equals  $n$ :

$$k_{+F}^2 + k_{-F}^2 = k_F^2 = 4\pi n \quad (221)$$

where  $k_F$  denotes the Fermi momentum of a fully polarized sea.

In the gapped case there were several critical fields  $B^c$ , each corresponding to one more CF-LL flipping its spin, each describing one more jump in the allowed values of  $P$ . In the gapless case the situation is different. For very large Zeeman energy, the sea will be fully polarized. It will not be worth including even one fermion of the opposite spin since the Zeeman energy cost alone will exceed the Fermi energy of the polarized sea. As we lower the Zeeman term, we will reach a critical field at which it will be worth introducing one fermion of the other spin with zero (effective) kinetic energy. At this point the energy of a particle on top of the spin-up sea obeys

$$\mathcal{E}_+(k_{+F}) = g \frac{e}{2m_e} \frac{B_{\perp}}{\cos \theta}. \quad (222)$$

If we lower the Zeeman term further, the polarization will fall continuously and be determined by  $\mathcal{E}_{\pm}(k_{\pm F})$ , the energies of the particles on top of these two seas according to

$$\mathcal{E}_+(k_{+F}) - \mathcal{E}_-(k_{-F}) = g \frac{e}{2m_e} \frac{B_{\perp}}{\cos \theta}. \quad (223)$$

This equation states that the system is indifferent to the transfer of a particle from one sea to another, i.e., has minimized its energy with respect to polarization.

Since the effective magnetic field vanishes at the gapless fractions, we deal with a very simple expression for  $\bar{\rho}(\mathbf{q})^p$ :

$$\bar{\rho}^p(\mathbf{q}) = \int \frac{d^2 k}{4\pi^2} (-2i) \sin\left(\frac{\mathbf{q} \times \mathbf{k} l^2}{2}\right) d_{\mathbf{k}-\mathbf{q}}^{\dagger} d_{\mathbf{k}}. \quad (224)$$

It is easy to do a HF calculation and obtain

$$\begin{aligned} \mathcal{E}_{\pm}(k) &= \\ & 2 \int \frac{d^2 q}{4\pi^2} \check{v}(q) \sin^2 \left[ \frac{\mathbf{k} \times \mathbf{q} l^2}{2} \right] \\ & - 4 \int \frac{d^2 k'}{4\pi^2} n_{\pm}^F(|k'|) \check{v}(|\mathbf{k} - \mathbf{k}'|) \sin^2 \left[ \frac{\mathbf{k}' \times \mathbf{k} l^2}{2} \right] \\ & \equiv \mathcal{E}_0 + \mathcal{E}_I \end{aligned}$$

where the Zeeman energy is not included,  $n_{\pm}^F$  is the (step) Fermi function for the two species,  $\mathcal{E}_0$  and  $\mathcal{E}_I$  represent single particle energy (due to what was called  $H_0$  earlier) and the energy of interaction of this particle at the Fermi surface with those inside the sea, and

$$\check{v}(k) = v(k) e^{-k^2 l^2 / 2}. \quad (225)$$

When this result is used to compute  $\mathcal{E}_+(k_{+F}) - \mathcal{E}_-(k_{-F})$ , we find once again that the numbers fit a free theory in the following sense. Imagine that CF were free and had a mass  $m_p$ . We would then have

$$\mathcal{E}_+(k_{+F}) - \mathcal{E}_-(k_{-F}) = \frac{k_{+F}^2 - k_{-F}^2}{2m_p} \quad (226)$$

What we find is that the HF number for  $\mathcal{E}(k_{+F}) - \mathcal{E}(k_{-F})$  may be fit very well to the above form with an  $m_p$  that is essentially constant as we vary  $k_{\pm F}$  i.e., the relative sizes of the up and down seas (which is analogous to an  $m_p$  that does not depend on the index  $r$  in the gapped case). This constant  $m_p$  defined by

$$\frac{1}{m_p} = 2 \frac{\mathcal{E}_+(k_{+F}) - \mathcal{E}_-(k_{-F})}{k_{+F}^2 - k_{-F}^2} \quad (227)$$

matches smoothly with that defined for the nearby gapped fractions.

This free-field behavior is surprising because we will shortly see that there are many reasons to believe that the CF's are not free. For now, note that the HF energies are not even quadratic in momenta: for example at  $\nu = \frac{1}{2}$  and  $\lambda = 1$  there is a hefty quartic term:

$$\frac{\mathcal{E}(k_{\pm F})}{(e^2/\varepsilon l)} = a \left( \frac{k_{\pm F}}{k_F} \right)^2 + b \left( \frac{k_{\pm F}}{k_F} \right)^4 \quad (228)$$

where  $a = .075$ ,  $b = -.030$ .

There is no reason the CF kinetic energy should be quadratic in momentum. These particles owe their kinetic energy to electron-electron interactions, and given this fact, all we can say is that their energy must be an even function of  $k$ , starting out as  $k^2$  at small  $k$ . What constitutes small  $k$  is an open question that is answered unambiguously here: our expression of the energy has substantial  $k^4$  terms for momenta of interest, but not from higher powers. An analogous result holds for the gapped fractions, where the CF-LL's are not equally spaced (Murthy 1999, Mandal and Jain, 2001a)

The proper interpretation of this free-field behaviour will be taken up next.

### C. Composite Fermions: Free at last?

The fact that magnetic phenomena at  $T = 0$  can be described (to excellent accuracy) by free fermions of mass  $m_p$  (Park and Jain, 1998, 1999, Shankar, 2000, 2001) needs to be properly understood and interpreted. In particular, one must resist the thought that perhaps by some further change of variables one could take the present hamiltonian and convert it to a free one. This is because if there were really an underlying free theory, it would have a single mass  $m_{CF}$  for both activation and polarization phenomena, with the CF's forming LL's of spacing  $eB^*/m_{CF}$ . But we know from the extended hamiltonian theory, Jain's approach, or experiment, that there are two masses  $m_a$  and  $m_p$  that differ by at least a factor of two. Furthermore the shape of the magnetoexciton dispersions (Kamilla and Jain, 1998, and Figure 5 also points to sizeable CF interactions: as  $ql^2$ , the distance between the particle and hole varies, the (binding) energy varies by an amount comparable to their individual energies, whose sum is given by the value at large  $ql$ .

It has been shown (Shankar, 2000, 2001) that a single assumption about the form of the ground state energy, an assumption that is not equivalent to the free-field assumption or even to a quadratic dispersion relation in the gapless cases, will explain this behavior for gapped and gapless fractions. Consider  $E(S)$ , the ground state energy as a function of  $S = nP$ , where  $P$  is the polarization. By rotational invariance it must have only even powers of  $S$  in its series. Assume the series is dominated by the first two terms:

$$E(S) = E(0) + \frac{\alpha}{2} S^2 \quad (229)$$

where  $\alpha$  is the inverse linear static susceptibility.

Consider first the gapless case. When  $dn$  particles go from spin-down to spin-up,

$$dE = \alpha S dS = \alpha S (2 dn) \quad (230)$$

$$= \alpha \frac{k_{+F}^2 - k_{-F}^2}{4\pi} (2 dn) \quad (231)$$

using the volumes of the Fermi seas. We see that  $dE$  has precisely the form of the kinetic energy difference of particles of mass  $m_p$  given by

$$\frac{1}{m_p} = \frac{\alpha}{\pi}. \quad (232)$$

Thus  $m_p$  is essentially the static susceptibility, which happens to have dimensions of mass in  $d = 2$ . The statement that  $m_p$  has no  $r$ -dependence in the gapped case or no spin dependence in the gapless case is the same as saying that the full nonlinear susceptibility does not depend on the spin  $S$ , which in turn means  $E(S)$  is quadratic in  $S$ .

Note that the free-field form of  $dE$  comes from  $E \simeq S^2$  and  $d = 2$ : in  $d = 3$ , we would have  $dE/dn \simeq S \simeq (k_{+F}^3 - k_{-F}^3)$  which no one would interpret as a difference of kinetic energies.

Let us see how this general argument applies to the specific example we have been working on.

Consider the HF energies quoted earlier

$$\frac{\mathcal{E}(k_{\pm F})}{(e^2/\varepsilon l)} = a \left( \frac{k_{\pm F}}{k_F} \right)^2 + b \left( \frac{k_{\pm F}}{k_F} \right)^4 \quad (233)$$

The quartic terms miraculously drop out in the energy cost of *transferring* a particle from the top of the spin-down sea to the top of the spin-up sea:

$$\frac{dE}{(e^2/\varepsilon l)} = a \frac{k_{+F}^2 - k_{-F}^2}{k_F^2} + b \frac{k_{+F}^4 - k_{-F}^4}{k_F^4} \quad (234)$$

$$= a \frac{k_{+F}^2 - k_{-F}^2}{k_F^2} + b \frac{(k_{+F}^2 - k_{-F}^2)(k_{+F}^2 + k_{-F}^2)}{k_F^4} \quad (235)$$

$$= \frac{(a+b)}{k_F^2} (k_{+F}^2 - k_{-F}^2) \quad (236)$$

using

$$k_{+F}^2 + k_{-F}^2 = k_F^2. \quad (237)$$

Note how  $d = 2$  was essential to this argument: in  $d = 3$  we would have  $k_{+F}^3 + k_{-F}^3 = k_F^3$ .

Thus the  $k^4$  terms in  $\mathcal{E}(k_{\pm})$  are not the cause of the  $S^4$  term. However, a  $k^6$  term in  $\mathcal{E}(k_{\pm})$ , can be shown to produce an  $S^4$  term in  $E(S)$ .

Thus the apparent free-field behavior is tied to the smallness of terms of order  $k^6$  and higher. To understand why the  $k^6$  term is so small, we turn to Eqn. (154) for  $H_0$ . Expanding the  $\sin^2$  in a series, we find the  $k^6$  term is down by a factor of at least 15 (50) relative to the  $k^2$  term, at  $\lambda = 0$  ( $\lambda = 1$ ), all the way up to  $k = k_F$ . Presumably this feature (and its counterpart in the gapped case) persists in the HF approximation to  $H$  and keeps  $E(S)$  essentially quadratic, which in turn mimics free-field behavior.

The reader is referred the original work (Shankar, 2001) for a proof that  $E(S) = E(0) + \frac{\alpha}{2} S^2$  implies that  $\Delta(r)$  will be  $r$ -independent in the gapped case as well.

Composite Fermions are not free fermions but are like Landau quasiparticles in a Fermi liquid<sup>40</sup>. These objects too are labeled by free-particle quantum numbers and long-lived. They do have fairly strong interactions: the dimensionless Landau parameters that describe these interactions are not small and produce effects like zero sound. They are adiabatically connected to free fermions in zero field just as the CF's are adiabatically connected to free fermions in a reduced field  $B^*$ .

#### D. Effective potentials for experimental systems with disorder

In comparing to experimental results one cannot neglect disorder. Our theory ignores disorder and our results are completely determined by the electron-electron interaction. Here we ask if it is possible that a ZDS potential with some effective  $\lambda$  can describe a dirty system. First of all, we realize this cannot be true with respect to all observables, if at all it is true for any. For example, if one were considering conductance, one knows the electron in a disordered potential will typically get localized whereas no ZDS interaction will predict this. As for transport gaps, the present day samples, with a disorder broadening of the same order as the gaps, again preclude this possibility. Magnetic transitions, on the other hand, are controlled by total energies and one may expect that disorder will have a rather innocuous effect and can be represented in an average way by some translationally invariant interaction. We raise this issue because in several magnetic phenomena to be described shortly, it appears that a single  $\lambda$  characterizes a sample. Specifically,  $\lambda$  extracted from one data point can be used to explain the rest of the data from that sample. If the other data points differ only in the temperature  $T$ , the same  $\lambda$  is used. If it differs in  $B$  or  $n$  or  $\nu$ , the following scaling argument applies (Ando, Fowler, and Stern, 1982): In a heterojunction, the donors of density  $n$  produce a confining linear potential of slope that goes as  $n$ . If one considers a variational wavefunction of the Fang-Howard (1966) form  $\psi(z) = A(w)z \exp(-z/w)$  in the transverse direction, then the optimal  $\bar{w}$  (to which  $\Lambda$ , the well width must be proportional), varies as  $\bar{w} \simeq n^{-\frac{1}{3}}$ . Consequently the dimensionless width,  $\lambda = \Lambda/l$  varies as

$$\lambda \simeq n^{-\frac{1}{3}} B^{\frac{1}{2}} \simeq B^{1/6} \nu^{-\frac{1}{3}} \simeq n^{1/6} \nu^{-\frac{1}{2}}. \quad (238)$$

Arguments can be given (Shankar 2001) for why in certain limiting conditions, not realized in today's experiments, an effective potential exists. The question of why this works in realistic situations far from this limit remains unanswered.

With this preamble, let us turn comparison with experimental results. Kukushkin, von Klitzing, and Eberl (1999) vary both  $n$  and  $B$  and drive the system through various transitions at  $T = 0$  (by extrapolation). The field  $B$  is

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<sup>40</sup> This was already suggested by Halperin, Lee, and Read (1993). Also in this context, see the work of Mandal and Jain (2001a).

TABLE V Critical fields based on a fit at  $\frac{3}{7}$ . The rows are ordered by the last column which measures density.

| $\nu$         | comment                     | $B^c$ (exp) | $B^c$ (theo) | $\nu B^c$ (exp) |
|---------------|-----------------------------|-------------|--------------|-----------------|
| $\frac{1}{4}$ | $(3, 1) \rightarrow (2, 2)$ | 2.7 T       | 1.6 T        | 1.2             |
|               | $(2, 0) \rightarrow (1, 1)$ | 3 T         | 2.65 T       | 1.2             |
|               | saturation                  | 5.2 T       | 4.4 T        | 1.3             |
|               | $(3, 0) \rightarrow (2, 1)$ | 4.5 T       | 4.5 T        | 1.93            |
|               | $(4, 0) \rightarrow (3, 1)$ | 5.9 T       | 5.9 T        | 2.62            |
|               | saturation                  | 9.3 T       | 11.8 T       | 4.65            |

always perpendicular to the sample. We will compare the hamiltonian theory to these experiments by calculating the critical fields at which the  $\nu = \frac{1}{2}$  and  $\nu = \frac{1}{4}$  systems saturate ( $P = 1$ ) and the gapped fractions undergo transitions from one quantized value of  $P$  to the next.

Let us recall that as far as these transitions go, the systems behave like free fermions of mass  $m_p$  which is independent of the index  $r$  which labels how many LL's have reversed their spins in the gapped case or the size of the up and down Fermi circles in the gapless cases.

We consider  $B^c$ 's at which the systems at  $1/4, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}$ , and  $\frac{1}{2}$  lose full polarization ( $r = 0$  for gapped cases, saturation for the gapless cases) and, for  $\frac{4}{9}$ , also the  $r = 1$  transition,  $|\mathbf{3}, \mathbf{1}\rangle \rightarrow |\mathbf{2}, \mathbf{2}\rangle$ .

An experimental complication needs to be addressed first. Each of these transitions seems to take place via a narrow intermediate step (Kukushkin, von Klitzing, and Eberl, 1999) with a polarization half-way between the ones allowed by CF theory based on spatially homogeneous states. We use the center of these narrow steps as the transition points for comparison to the present theory. The physics of these intermediate steps will be addressed in Section X.

In accordance with our strategy we find  $\lambda$  by fitting the theory to the experiment for a particular transition. We will use this value (or for samples with changing field and density, Eqn. (238)) to predict  $B^c$  for other transitions using Eqn. (220). We obtain  $\lambda_{\frac{3}{7}} = 1.42$  from the transition  $|\mathbf{3}, \mathbf{0}\rangle \rightarrow |\mathbf{2}, \mathbf{1}\rangle$  at  $B^c = 4.5T$ .

For the gapless cases, there are two equivalent approaches. First, at the critical field the Fermi energy of the up spins equals the Zeeman energy of the down spins:

$$g \left[ \frac{eB^c}{2m_e} \right] = \frac{k_F^2}{2m_p} = \frac{2\pi n}{m_p} = \frac{eB\nu}{m_p} \quad (239)$$

Equivalently we can write for the total ground state energy density  $E^Z(S)$ , (where the superscript indicates that the Zeeman energy is included),

$$E^Z(S) = \frac{\alpha}{2} S^2 - g \frac{e}{2m_e} \frac{B_{\perp} S}{\cos \theta} \quad (240)$$

where  $\alpha = \pi/m_p$ . This expression is minimized (for  $P \leq 1$ ) to give  $P$ . Setting  $P = 1$  gives the critical fields.

The comparison to experiment is made in Table (V). Note that in rows above (below)  $\frac{3}{7}$ , where we fit  $\lambda$ , the predicted  $B^c$ 's are lower (higher) than the observed values, i.e., the actual  $\lambda$ 's are less (more) than what Eqn. (238) gives. This is consistent with the expectation that interactions will increase the effective thickness with increased density. If we fit to the  $\frac{2}{5}$  point, we obtain similar numbers, with the agreement worsening as we move off in density from  $\frac{2}{5}$ . Thus  $\frac{3}{7}$  was chosen as the fitting point since its density was somewhere in the middle of all the densities considered.

## IX. PHYSICS AT NONZERO TEMPERATURES $T > 0$ .

So far we have seen that the EHT may be used to compute quantities such as gaps, particle-hole profiles, critical fields for magnetic transitions and so on to 10-20% accuracy. All such quantities have been readily computed using trial wavefunctions, giving numbers that are superior to ours. Our main emphasis has been to expose the underlying physics as transparently as possible and to resolve questions such as why CF's behave like free particles on some occasions.

We turn to physics at finite  $T$  (Murthy, 2000c, Shankar, 2000, 2001) where the hamiltonian method has few rivals. Exact diagonalization (Chakraborty and Pietiläinen, 1996, Chakraborty, Niemelä, and Pietiläinen, 1998) is limited to very small systems and trial wavefunctions typically cover the ground state and very low-energy excitations. The hamiltonian approach is able to yield the temperature dependence of polarization  $P$  and the relaxation rate  $1/T_1$  for the gapless states in the thermodynamic limit. If  $\lambda$  is treated as before (fit to one data point per sample) we will see it is possible to give a very satisfactory account of experiments in gapless systems up to about  $1^{\circ}K$ , which is of the order of the Fermi energy (Shankar, 2000, 2001).



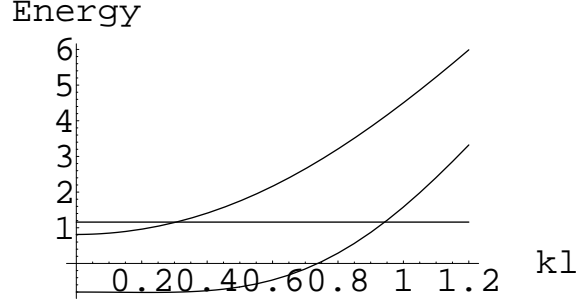


FIG. 9 Hartree Fock energies at  $\nu = \frac{1}{2}$  for up and down spins (upper and lower curves) at  $T = .3^\circ K$  at  $B = 5.52$  and zero tilt. Note that they are not simply quadratic in momenta, and that at the chemical potential, indicated by the horizontal line, the two graphs have very different slopes, i.e., density of states.

We then address finite-temperature polarization in gapped states, which is complicated by a nonzero spontaneous polarization and the attendant spin waves. It turns out to be essential to take the finite- $T$  behavior of these spin-waves into account. Once this is done, the theoretical predictions are in excellent agreement with the experiment up to several Kelvin (Murthy, 2000c).

We start with the gapless case since it is simpler.

#### A. Polarization and relaxation in gapless states

The polarization  $P$  is computed as follows. First we compute the HF energy of a particle *including the Zeeman energy*, which is the self-consistent solution to

$$\begin{aligned} \mathcal{E}_{\pm}^Z(k) = & \mp \frac{1}{2}g \left[ \frac{eB}{2m} \right] + 2 \int \frac{d^2q}{4\pi^2} \tilde{v}(q) \sin^2 \left[ \frac{\mathbf{k} \times \mathbf{q} l^2}{2} \right] \\ & - 4 \int \frac{d^2k'}{4\pi^2} n_{\pm}^F(|k'|) \tilde{v}(|\mathbf{k} - \mathbf{k}'|) \sin^2 \left[ \frac{\mathbf{k}' \times \mathbf{k} l^2}{2} \right] \end{aligned}$$

where the superscript on  $\mathcal{E}_{\pm}^Z$  reminds us it is the total energy including the Zeeman part, the Fermi functions

$$n_{\pm}^F(|k|) = \frac{1}{\exp[(\mathcal{E}_{\pm}^Z(k) - \mu)/kT] + 1} \quad (241)$$

depend on the energies  $\mathcal{E}_{\pm}^Z(k)$  and the chemical potential  $\mu$ . Fig. 9 shows some typical results. At each  $T$ , one must choose a  $\mu$ , solve for  $\mathcal{E}_{\pm}^Z(k)$  till a self-consistent answer with the right total particle density  $n$  is obtained. From this one may obtain the polarization by taking the difference of up and down densities. As usual we use the ZDS potential for which

$$\tilde{v}(q) = e^{-q^2 l^2 / 2} \frac{2\pi e^2 e^{-q l \lambda}}{q}. \quad (242)$$

The computation of the longitudinal nuclear relaxation rate  $1/T_1$  is more involved (Shankar, 2000, 2001). The question we ask is the following. The fermions are in a quantum well, with their density varying across the width. So the nuclear relaxation rate will be a function of position. Consider a nucleus at the center of the quantum well, (as well as the  $x - y$  plane) where the density is the largest. Let us call this point the origin and let  $1/T_1$  be the relaxation rate here. The theory predicts

$$\begin{aligned} \frac{1}{T_1} = & 4\pi k_B T \left( \frac{K_{\nu}^{max}}{n} \right)^2 \\ & \times \int_{E_0}^{\infty} dE \left( \frac{dn^F(E)}{dE} \right) \rho_+(E) \rho_-(E) F(k_+, k_-) \end{aligned} \quad (243)$$

$$F = e^{-(k_+^2 + k_-^2) l^2 / 2} I_0(k_+ k_- l^2) \quad (244)$$

$$\rho_{\pm}(E) = \int \frac{k dk}{2\pi} \delta(E - \mathcal{E}_{\pm}^Z(k)) \quad (245)$$

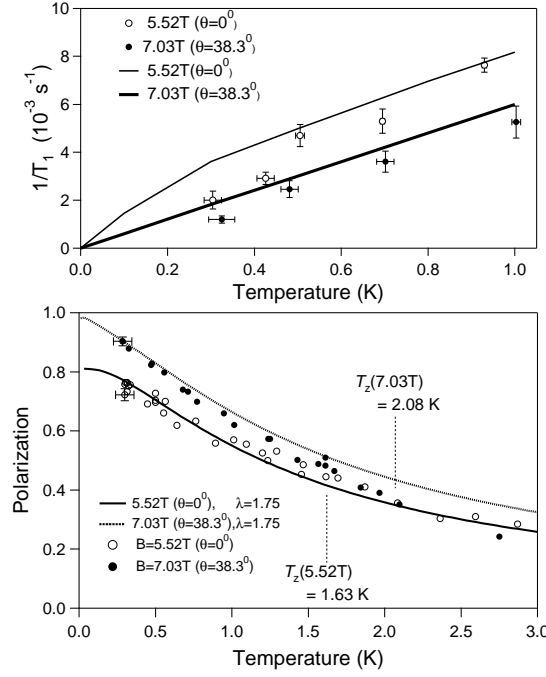


FIG. 10 Comparison to the work of Dementyev *et al.* The value of  $\lambda$  is fit to  $P$  at 300 mK,  $B_{\perp} = 5.52$  T and the rest follows from the theory. Notice the correlation between the curvature of  $1/T_1$  and the limit of  $P$  as  $T \rightarrow 0$  K.

where  $k_{\pm}$  are solutions to  $\mathcal{E}_{\pm}^Z(k_{\pm}) = E$ ,  $I_0$  is the Bessel function,  $E_0$  is the lowest possible energy for up spin fermions, and  $K_{\nu}^{max}$  is the measured maximum Knight shift (at the center of the sample) for the fraction  $\nu = \frac{1}{2}$  or  $\frac{1}{4}$ .

Here is a rough description of the derivation, the details of which may be found in Shankar (2000). Suppose for a moment we were dealing with electrons and not CF's. The Knight shift at the chosen point, the origin, will be determined by the spin density there. The same parameter enters the  $1/T_1$  calculation quadratically. This is why  $K_{\nu}^{max}$  enters the answer. Thus  $K_{\nu}^{max}$  is not calculated *ab initio* but taken from the same experiment. The density of states and Fermi factor are standard. The only new feature here is the presence of  $F(k_+, k_-)$  which reflects the fact that the spin density has to be projected into the LLL when going to the CF basis. The effect of this factor (which is none other than the  $e^{-q^2 l^2/2}$  which appeared on the projected charge density) is to suppress processes with momenta much larger than  $1/l$ , as these have no place within the LLL.

We now compare to some experiments at  $\nu = \frac{1}{2}$  and  $T > 0$ . Consider first Dementyev *et al* (1999). From their data point  $P = .75$  for  $B = B_{\perp} = 5.52$  T at 300 mK We deduce

$$\lambda = 1.75. \quad (246)$$

We have once again chosen to see to what extent a *sole* parameter  $\lambda$ , can describe  $P$  and  $1/T_1$  for the given sample at a given  $B_{\perp}$ , but various temperatures and tilts.

Since there does not exist a model, including disorder, that describes how  $\lambda$  should vary with tilt we include no such variation.

Dementyev *et al* (1999) find  $K_{\frac{1}{3}}^{max} = 4.856 \cdot 10^{-7} K$ , which is believed to describe a saturated system at  $P = 1$ . They estimate that  $K_{\frac{1}{2}}^{max} = .953 K_{\frac{1}{3}}^{max}$ , which is what we need here. Given this information,  $1/T_1$  follows.

The top and bottom halves of Figure 10 compare the HF calculation of  $1/T_1$  and  $P$  respectively, to the data. (The graphs for  $1/T_1$  differ slightly from those in Shankar (2000) since the present calculation treats the spin of the CF more carefully. The  $1/T_1$  graph at 5.52 T appears a little jagged since it was computed at just six points which were then connected. This is not apparent in the tilted case since the points lie on a straight line.

Dementyev *et al* (1999) had pointed out that a two parameter fit (using a mass  $m$  and interaction  $J$ ), led to disjoint sets of values (in the  $m - J$  plane) for these four curves. Given that  $H$  is neither free nor of the standard form  $(p^2/2m + V(x))$  this is to be expected. *By contrast, a single  $\lambda$  is able to describe the data here rather well since  $H$  has the right functional form.* Given how the theory fits the polarization data up to the Fermi energy of  $\simeq 1^\circ K$ , it is clear that changing the data point used to fix  $\lambda$  will be inconsequential.

The present work establishes a phenomenological, nontrivial, and nonobvious fact that a single  $\lambda$  parameter, determined from one data point, can describe both  $P$  and  $1/T_1$  for the given sample under a variety of conditions. That the fitted  $\lambda$  is larger than the Local Density Approximation value makes sense, as both disorder and LL mixing will lower the gap and raise  $\lambda$ .

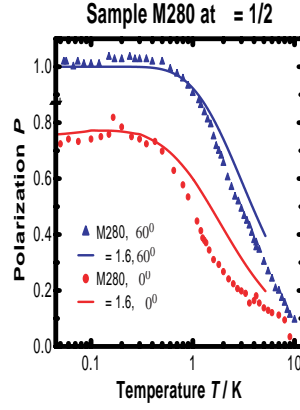


FIG. 11 Comparison to Melinte *et al* at  $\theta = 0$  and  $60^\circ$  with  $\lambda = 1.6$  fit to  $P$  at 60 mK,  $B_\perp = 7.1$  T.

Consider next sample M280 of Melinte *et al* (2000) which had  $P = .76$  at  $.06^\circ K$  and  $B = B_\perp = 7.1T$ , from which we deduced  $\lambda = 1.6$ . Figure 11 compares the theoretical  $T$ -dependence of  $P(T)$  with data. There is a factor of 2 between theory and experiment for  $1/T_1$ , not shown.

The fair agreement for the tilted cases is unexpected in both the Dementyev *et al* (1999) and Melinte *et al* (2000) data. First of all, orbital effects have to be considered due to the tilt. The thickness parameter  $\Lambda$  can be affected by it. As pointed out by Jungwirth in a private communication, once there is an in-plane component of  $B$ , the problem is no longer rotationally invariant. This means that our states are no longer HF states and get scattered into each other by the potential. No attempt is made here to take into account all the effects of the tilt. Instead we included just the increased Zeeman coupling and hoped for the best.

For the benefit of others who measure  $1/T_1$  at  $\nu = \frac{1}{2}$  in the future on similar samples, we give some very approximate formulae (to be used for zero or small tilts). From Figure 10, we note that in general, the graphs of  $1/T_1$  become linear and parallel for temperatures above  $.3^\circ K$ . In this region we can write

$$\frac{d(1/T_1)}{dT} \simeq 3 \left[ \frac{\bar{K}}{\bar{n}} \right]^2 \cdot 10^{-3} s^{-1} [^\circ K]^{-1} \quad \text{for } T > .3^\circ K \quad (247)$$

with  $\bar{K}$  the Knight shift in KHz and

$$\bar{n} = \frac{n}{10^{10}/cm^2} \quad (248)$$

(In this approximate formula, we ignore the  $\lambda$  dependence of Eqns. (243-245), and the distinction between the average and maximum Knight shift.)

The graphs do not generally obey the Korringa-like law because as  $T \rightarrow 0$  they are sublinear (superlinear) for saturated (unsaturated) cases. Only the critical case with  $P(0) \rightarrow 1$  as  $T \rightarrow 0$  is linear. For  $T > .3^\circ K$ , (which in general must be replaced by either the energy gap or energy overlap between the up and down Fermi energies) we have the approximate result

$$\frac{1}{T_1} \simeq [3 T(^\circ K) + C] \left[ \frac{\bar{K}}{\bar{n}} \right]^2 \cdot 10^{-3} s^{-1} \quad (249)$$

$$C = 0 \quad (\text{critical}) \quad (250)$$

$$= > 0 \quad (\text{unsaturated}) \quad (251)$$

$$= < 0 \quad (\text{saturated}) \quad (252)$$

For the critical case (only), we have a Korringa law

$$\frac{1}{T_1 T(^\circ K)} \simeq [3] \left[ \frac{\bar{K}}{\bar{n}} \right]^2 \cdot 10^{-3} s^{-1} (^\circ K)^{-1} \quad (253)$$

For Dementyev *et al* (1999)  $C \simeq 1$ . This value may be used as a first approximation. For more accurate results one must solve Eqns. (243-245).

## B. Polarization in gapped states

Let us now turn to the theoretical description of the finite-temperature polarization of the  $\nu = \frac{1}{3}$  state, detailed measurements of which have recently been carried out (Khandelwal *et al*, 1998, Melinte *et al*, 2000), Freytag (2001). Theoretical details can be found in Murthy (2000c).

While the CF-HF approximation provided an adequate description of  $\nu = \frac{1}{2}$  it is *qualitatively* incorrect for a spontaneously polarized state like  $\nu = \frac{1}{3}$ . The reason is that it underestimates the effects of the excitations that destroy order. For example, it predicts a nonzero spontaneous polarization at  $E_Z = g\mu B_{tot} = 0$  and  $T > 0$ , in violation of the Hohenberg-Mermin-Wagner theorem (Mermin and Wagner, 1966, Hohenberg, 1967) that forbids spontaneous breaking of a continuous symmetry in two dimensions, except at  $T = 0$ . The  $T = 0$  spontaneous polarization is driven by the fact that fermions of the same spin will avoid each other due to the Pauli principle and thus have a lower interaction energy, just as in the  $\nu = 1$  spontaneous quantum ferromagnet<sup>41</sup>, which has been extensively studied theoretically<sup>42</sup> and experimentally<sup>43</sup>.

Thus we must begin by understanding the disordering mechanism and seeking a proper description of it. Consider a fully polarized  $\nu = \frac{1}{3}$  state at  $T = 0$  with all the fermions in the CF-LLL state. As the system is heated, some fermions will go to the  $n = 1$  CF-LL with spin up (which does not change the polarization) and some will go to spin down  $n = 0$  CF-LL, which reduces the polarization and costs an energy per spin-flip of  $\Delta_{SR}$ , the *spin-reversed gap*.

This description completely misses the spin-waves which are related to the particle-hole excitations as follows. Just as in the case of the magnetoexciton (Figure 4) where no spin was flipped, a spin-flip particle-hole excitation has an energy that varies with  $q$ <sup>44</sup>. As  $q \rightarrow \infty$ , the dispersion settles down at  $\Delta_{SR}$ , the energy to create a widely separated pair. However as  $q \rightarrow 0$  the pair energy vanishes, for  $E_Z = 0$ , by Goldstone's theorem. For nonzero  $E_Z$  the long wavelength limit for the spin-wave energy is  $E_Z$  by Larmor's theorem. These are the modes to reckon with for they are very low in energy and plentiful at low temperatures.

The simplest way to describe these spin-waves (including their self-interaction) is the continuum quantum ferromagnet (CQFM)(Read and Sachdev, 1996). This model assumes all high-energy modes (at the electronic and CF cyclotronic scales) have been integrated out. The only modes left are slow and long-wavelength fluctuations of the spin polarization, which have the action

$$S = \int d^d x \int_0^{1/T} d\tau (iM_0 \mathbf{A}(\mathbf{n}) \cdot \nabla_\tau \mathbf{n} + \frac{\rho_s}{2} (\nabla_x \mathbf{n})^2 - M_0 \mathbf{H} \cdot \mathbf{n} + \dots) \quad (254)$$

where  $M_0$  is the magnetization density,  $\mathbf{n}$  is a local vector of unit length pointing in the direction of the magnetization,  $\mathbf{A}(\mathbf{n})$  is the field that implements the Berry's phase needed to obtain the correct quantum commutation relations between the spin components,  $\rho_s$  is the spin stiffness, and  $\mathbf{H} = g^* \mu_B \mathbf{B}$  is the Zeeman field ( $|\mathbf{H}| = E_Z$ ).

This model is still nontrivial (because of the condition  $|\mathbf{n}|^2 = 1$ ). However it can be solved (Read and Sachdev, 1996) in the limit when  $N \rightarrow \infty$ , where  $N$  is the number of components of  $\mathbf{n}$ . The limit  $N \rightarrow \infty$  appears to describe the actual case of  $N = 3$  in the case  $\nu = 1$ .

Our strategy then will be deduce reasonable values for  $M_0$ , the magnetization density, and  $\rho_s$ , the spin stiffness, and then plug them into the known results from the large- $N$  limit for the magnetization as a function of  $T$  is given by

$$P(T) = M_0 \Phi_M(r, h) \quad (255)$$

where  $r = \rho_s/T$  and  $h = E_Z/T$  are scaling variables, and  $\Phi_M$  is a known scaling function (Read and Sachdev, 1996)<sup>45</sup>.

To find the values of the parameters  $M_0$  and  $\rho_s$  corresponding to the  $\nu = \frac{1}{3}$  state, we will use some Hartree-Fock results. Since the underlying fermionic theory responds to temperature by self-consistently modifying occupations and energies, we expect to obtain temperature-dependent values<sup>46</sup>  $M_0(T)$  and  $\rho_s(T)$ .

<sup>41</sup> The classic study of nontrivial interaction effects (beyond ferromagnetism) at  $\nu = 1$  is Sondhi, Karlhede, Kivelson, and Rezayi (1993).

<sup>42</sup> Kopietz and Chakravarty (1989), Read and Sachdev (1996), Kasner and MacDonald (1996), Haussmann (1997), Timm, Girvin, Henelius, and Sandvik (1998), Kasner, Palacios, and MacDonald (2000),

<sup>43</sup> Tycko, Barrett, Dabbagh, Pfeiffer, and West, (1995), Barrett, Dabbagh, Pfeiffer, West, and Tycko, (1995), Aifer, Goldberg, and Broido (1996), Manfra, Aifer, Goldberg, Broido, Pfeiffer, and West (1996).

<sup>44</sup> For an example of this calculation see Murthy (1999)

<sup>45</sup> There are actually two different large- $N$  approximations corresponding to the fact that the symmetry group can be viewed as an example of  $O(N)$  with  $N = 3$ , or as an example of  $SU(N)$  with  $N = 2$ . Both are considered by Read and Sachdev (1996) and scaling functions given.

<sup>46</sup> This should be contrasted with the case  $\nu = 1$  where due to the huge exchange gap, particle-hole excitations are frozen at all temperatures of interest, and the parameters of the CQFM are  $T$ -independent.

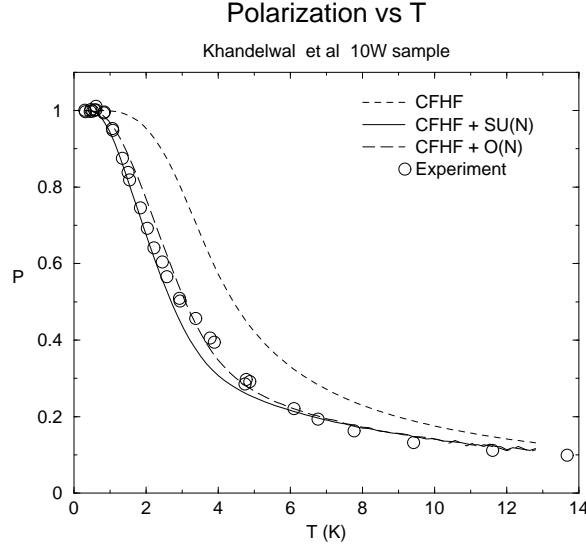


FIG. 12 Polarization  $P(T)$  as a function of  $T$  for  $\frac{1}{3}$  for the data of Khandelwal *et al*'s (1998) 10W sample. The thickness parameter has been set to  $\lambda = 1.5$ . Note the important role played by the spin-waves in bringing the HF value of the polarization down to the nearly experimental value.

First consider the spin stiffness  $\rho_s(T)$ . At a given temperature  $T$  the self-consistent occupations  $N_{F,GS}(\sigma, n)$  and energies  $\epsilon(\sigma, n)$  in the ground state are computed using the procedure described in the gapless case. Now one creates a twisted spin state and computes the HF energy of the twisted ground state (Murthy, 2000c), and thence the excess energy to order  $q^2$ . Comparing to the energy cost of a twist in the CQFM, which is  $(\rho_s/2)L^2q^2$ , one finds the spin stiffness

$$\rho_s = \frac{1}{16\pi} \int \frac{d^2s}{(2\pi)^2} v(s) \sum_{n_1, n_2} |\rho_{n_1 n_2}(s)|^2 \times (N_F(\uparrow, n_1) - N_F(\downarrow, n_1))(N_F(\uparrow, n_2) - N_F(\downarrow, n_2)) \quad (256)$$

where  $L^2$  is the area of the system, and  $\tilde{\rho}_{n_1 n_2}$  is the matrix element of equation (D1). The above should be regarded as an estimate for the twist rather than a rigorous calculation (even in HF), since ideally one should compute the free energy cost of a twist, rather than just the internal energy cost, as we have done.

To find  $M_0(T)$  we need a more devious approach (Murthy, 2000c), based on Eqn. (255). We already know how to compute the CF-HF magnetization  $P_{HF}(T)$ . We set

$$P_{HF}(T) = M_0(T) \Phi_M(r=0, h=\Delta_{SR}/T) \quad (257)$$

and justify it as follows. In the CF-HF theory the particles and holes are treated as independent, or noninteracting, with a gap equal to  $\Delta_{SR}$  independent of the distance between them. This corresponds to a collective mode dispersion that is completely flat,  $\omega(q) = \Delta_{SR}$ . The CQFM description that corresponds most closely to the CF-HF is the one that has the same spin-flip excitation spectrum, namely one with no spin stiffness ( $r=0$ ), and an effective Zeeman field  $E_Z^{eff} = \Delta_{SR}$ .

Armed with  $M_0(T)$  and  $\rho_s(T)$ , we evaluate the large- $N$  scaling functions and hence  $P(T)$ . The predictions are shown in Figure 12 for parameters corresponding to the experiment of Khandelwal *et al* (1998). As can be seen, CF-HF overestimates the spin magnetization considerably at low temperatures, but the inclusion of interacting spin-waves gives a prediction in almost perfect agreement with the data. The results are quite insensitive to  $\lambda$  and a typical value of  $\lambda = 1.5$  was used. The results are also insensitive to which of the two large- $N$  approximations is used. Similar good agreement over a wide range of temperatures is found in the comparison to the data of Melinte *et al*'s (2000) M242 sample, shown in Figure 13. Here we have plotted the Knight shift vs. the temperature. The Knight shift at very low temperature shows considerable scatter, and an appropriate intermediate value has been used to fit to the theory.

It is somewhat surprising that the theory agrees so well with the data up to temperatures much higher than those for which agreement was found for the gapless fractions. We expect the theory to work only when CF's are well-defined, which should be true up to temperatures of the order of the  $\frac{1}{3}$  gap. In fact, the agreement persists to about 10K, which is higher than a typical activation gap for  $\frac{1}{3}$  in such samples. For more details see Murthy (2000c).

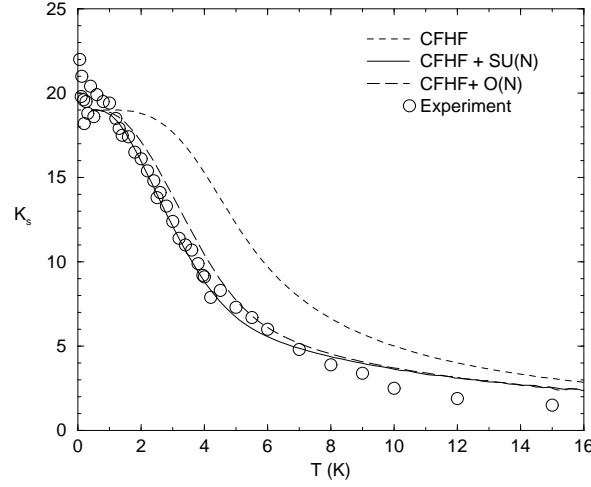


FIG. 13 Polarization  $P(T)$  as a function of  $T$  for  $\frac{1}{3}$  for the data of Melinte *et al's* (2000) M242 sample. The thickness parameter has again been set to  $\lambda = 1.5$ . Once again spin fluctuations are seen to be crucial in reproducing the experimental polarization.

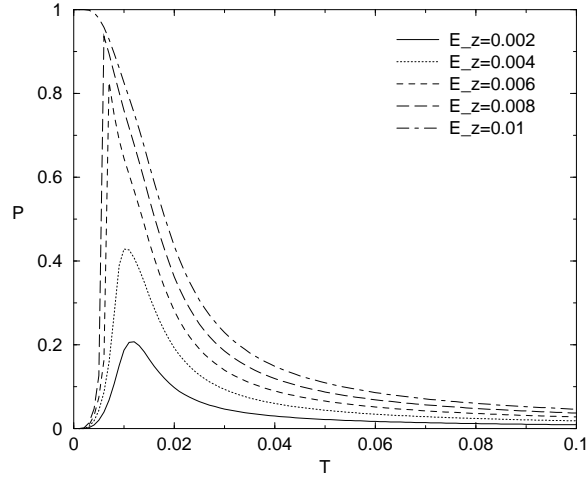


FIG. 14 Polarization as a function of  $T$  for  $\frac{2}{5}$ .

The same considerations can be applied to the  $\nu = \frac{2}{5}$  state. Since it is unpolarized for  $E_Z = 0$  we can stop at the CF-HF stage. Figure 14 shows the  $P(T)$  curves for  $\nu = \frac{2}{5}$  for  $\lambda = 1.5l$  for a range of Zeeman couplings. The rise and fall of  $P(T)$  at small  $E_Z$  can be understood as follows. At  $T = 0$ , the system is in a singlet state with the up and down spins occupying the CF-LLL. As the system is heated, some particles go the next LL. Here the ones with spin parallel to the external field are preferred due to the Zeeman term and hence a nonzero polarization develops. At higher  $T$ , the two spin species start to get occupied with nearly equal probabilities and  $P$  starts to decline. There is a transition to the fully polarized state around  $E_Z = 0.01E_C$ .

One may compute  $P(T)$  for arbitrary fractions by exact diagonalization (keeping all the excited states) and subsequent calculation of thermodynamic quantities (Chakraborty and Pietiläinen, 1996). Due to computational limitations, this method is restricted to fairly small systems. For example, the largest system studied by Chakraborty and Pietiläinen (1996) for  $\nu = \frac{1}{3}$  has 5 electrons, and for  $\nu = \frac{2}{5}$  has 4 electrons. Allowing for this, the results seem fairly consistent with ours. Currently experimental data are not available for comparison.

Finally one use the CS theory to do finite  $T$  calculations, using  $m^*$  as a free parameter instead of  $\lambda$ .

## X. INHOMOGENEOUS STATES OF CF'S

We have repeatedly emphasized that CF's are interacting particles. The uniform liquid states studied so far were not contingent on CF interactions— such states could exist even for noninteracting CF's. We now turn to some interesting inhomogeneous states, which would not exist if the CF's were free, such as the high-field Wigner Crystal,

and possibly the partial polarized states seen in the gapped fractions by Kukushkin, von Klitzing, and Eberl (1999).

### A. The high-field Wigner Crystal

The clearest instance of an inhomogeneous state in the fractional quantum Hall regime is the high-field Wigner crystal (WC) (Yoshioka and Fukuyama, 1979). If we start with a free electron gas and slowly crank up the interactions, or lower the density, the emphasis goes from the kinetic to potential energy. In the Wigner crystal the electrons seek an arrangement aimed at minimizing the potential energy. In the Hall case, due the quenching of kinetic energy, the WC seems even more likely at low densities. It was initially proposed (Fukuyama and Platzman, 1982) as a possible explanation of the fractional quantum Hall effect. However, it was soon realized (Yoshioka and Lee, 1982) that this state is very different from the fractional quantum Hall states, because both its longitudinal and Hall conductances vanish at zero temperature and when the crystal is pinned by disorder (Yoshioka 1983, MacDonald 1984), leading to an insulating state. Further, the WC state is not tied to any particular commensurate filling.

There are now experimental observations<sup>47</sup> which support the existence of the WC near  $\nu = \frac{1}{5}$ . In fact, experiments see a *re-entrant* transition (Jiang *et al*, 1990, Jiang *et al*, 1991) in which there is a putative WC both above and below  $\frac{1}{5}$ . To see how this might happen, recall that the incompressibility of the Laughlin state results in a downward cusp in the ground state energy as a function of filling factor in the neighborhood of (say)  $\frac{1}{5}$ . Thus, one can easily imagine that in a small neighborhood of  $\frac{1}{5}$  the Laughlin liquid has a lower energy than the WC, leading to the re-entrant WC near  $\frac{1}{5}$ .

Theoretical work on the WC<sup>48</sup>, based on a study of trial wavefunctions and collective excitations has established that the Laughlin state becomes unstable around  $\nu \approx \frac{1}{6}$  to a density wave even in the absence of disorder. In the wavefunction approach, Hartree-Fock and (weakly) correlated wavefunctions have also been written down, and their energy evaluated. By studying the excitonic instabilities of the Laughlin liquid, Jain and Kamilla (1998) showed that it becomes unstable to crystallization around  $\nu = \frac{1}{9}$ .

Let us begin with one of the simplest HF wavefunctions for the crystal, that of Maki and Zotos (1983):

$$\Psi_{HF}(\{\mathbf{r}_i\}) = \mathcal{A} \prod_i \phi_{\mathbf{R}_i}(\mathbf{r}_i). \quad (258)$$

where  $\mathcal{A}$  is the antisymmetrization operator, and  $\phi_{\mathbf{R}_i}$  is a single-particle wavefunction that is localized at  $\mathbf{R}_i$  (lattice site) and belongs to the LLL. It is given by

$$\phi_{\mathbf{R}_i}(\mathbf{r}) = e^{-|\mathbf{r}-\mathbf{R}_i|^2/4l_0^2 - i\mathbf{r} \times \mathbf{R}_i \cdot \hat{z}/2l_0^2}. \quad (259)$$

The wavefunction (258) can be improved by adding a correlation factor corresponding to including small fluctuations around the HF state. The energy of this state becomes lower than that of the liquid state at about the experimentally right filling fraction ( $\nu \approx \frac{1}{7}$ ) (Maki and Zotos, 1983, Lam and Girvin, 1984, Levesque, Weiss, and MacDonald, 1984).

Experiments (Jiang *et al*, 1990, Jiang *et al*, 1991) show transport gaps two orders of magnitude smaller than the theoretical estimate as calculated using the Hartree-Fock approximation (Fukuyama, Platzman, and Anderson, 1979, Yoshioka and Fukuyama, 1979). The measurements of the Hall resistivity  $\rho_{xy}$  are surprising as well (Goldman, Wang, Su, and Shayegan, 1993, Goldys *et al*, 1992). As mentioned above, the WC is expected to have a vanishing Hall conductance  $\sigma_{xy} = 0$  (when pinned), which implies a vanishing Hall resistance  $\rho_{xy} = 0$ . However, the experiments see “Hall insulating” behavior (Zhang, Kivelson, and Lee 1992), that is,  $\rho_{xy} \approx \frac{2\pi\hbar}{\nu e^2}$ . These problems led Yi and Fertig (1998) to consider crystalline states with correlation zeroes that keep electrons apart. Each electron is combined with  $2s$  vortices to obtain the trial wavefunction

$$\Psi(\{\mathbf{r}_i\}) = \mathcal{A} \prod_{i \neq j} (z_i - z_j)^{2s} \prod_i \phi_{\mathbf{R}_i}(\mathbf{r}_i). \quad (260)$$

They also find that near  $1/5$  the best energies are obtained by attaching 4 zeroes to each electron. In other words, they take the Laughlin FQHE wavefunction at  $\frac{1}{5}$  which is a product of a Jastrow factor with quartic zeros and a single filled CF Landau level  $\chi_1$ , and replace  $\chi_1$  with a crystal. The Coulomb energy for this wavefunction is computed using

<sup>47</sup> Jiang, Willett, Störmer, Tsui, Pfeiffer and West (1990), Jiang, Störmer, Tsui, Pfeiffer and West (1991), Engel, Sajoto, Li, Tsui and Shayegan (1992), Goldman, Wang, Su, and Shayegan (1993), Goldys, Brown, Dunford, Davies, Newbury, Clark, Simmonds, Harris and Foxon (1992). For a recent review of the experimental situation, see Shayegan (1997)

<sup>48</sup> Maki and Zotos (1983), Lam and Girvin (1984), Levesque, Weiss and MacDonald (1984), Price, Platzman, and He (1993). For a recent review of the theory, see Fertig (1997).

Monte Carlo methods. Yi and Fertig (1998) have shown that the ground state energy of the correlated WC is lower than that of the usual WC at experimentally relevant filling fractions. Moreover by introducing Laughlin-Jastrow correlations between the interstitials and the lattice electrons the experimentally observed  $\rho_{xy}$  (Zheng and Fertig, 1994a, 1994b) can be explained. Unfortunately, the method becomes too computationally demanding to allow one to calculate other quantities of interest, such as the excitation spectrum.

This is a situation tailor-made for the EHT. Before we describe the theory and its results, let us note that there are two features of the experiment which hint that a CF-WC is involved. The first is the nonmonotonic behavior of the gap near  $\frac{1}{5}$ , which shows that the  $\frac{1}{5}$  Laughlin liquid correlations are being felt in the nearby WC state. Secondly, the threshold electric field beyond which the WC becomes depinned and starts sliding *increases* near  $\frac{1}{5}$ . Below we will see its connection with the structure and properties of the CF-WC.

Now let us see how to set up the EHT. Since attaching the zeros to electrons converts them into CF's, we are naturally led to consider a WC of CF's. Here is how one proceeds (details can be found in Narevich, Murthy, and Fertig, 2001). The crystal is characterized by density wave order parameters at the reciprocal lattice vectors  $\mathbf{G}$ .

$$\Delta_{nn'}(\mathbf{G}) = \frac{2\pi(l^*)^2}{L^2} \sum_X e^{-i\mathbf{G}_X \cdot \mathbf{X}} < d_{n,X-\mathbf{G}_Y l^{*2}/2}^\dagger d_{n',X+\mathbf{G}_Y l^{*2}/2} > \quad (261)$$

First, one has to assume a particular lattice structure (shape and size). An important parameter of lattice is the number of quanta of effective flux that penetrate each unit cell<sup>49</sup>. Things become simple when this number is rational, of the form  $p/q$ , where  $p$  and  $q$  have no common factors. In this case each CF-LL breaks up into  $p$  sub-bands with the total number of states in the original CF-LL being equally divided among the sub-bands (Yoshioka and Fukuyama, 1979). While the original CF-LL had a sharp energy, the sub-bands have a nonzero energy dispersion, which can be found from the HF hamiltonian. One finally closes the circle by demanding self-consistency; the ground state formed by filling up the sub-bands with the correct number of particles should reproduce the assumed form of Eqn. (261). Once one has a self-consistent HF solution, the transport gap is found as the energy difference between the centers of the highest occupied sub-band and the lowest unoccupied sub-band. The WC is characterized by one particle per unit cell, and depending on the filling, this translates to different  $p$  and  $q$ .

Here are the results: The gaps calculated from the Hamiltonian theory are within a factor of two of the experimental results, (Figure 15) which is to be contrasted to the more than two orders of magnitude disagreement with electronic HF calculations. The theory also predicts that the gap below  $1/5$  should be substantially more than the gap between  $1/5$  and  $2/9$  (the next quantum Hall liquid state). Narevich, Murthy, and Fertig (2001) are not able to go very close to  $1/5$ , because then  $p$  and  $q$  become large, and the problem becomes computationally prohibitive.

Two other results emerge at a qualitative level. First, the shear modulus becomes very small as  $\frac{1}{5}$  is approached, which means that the crystal becomes very soft to deformations. The reason is that as  $\frac{1}{5}$  approaches the WC tends more and more to a Laughlin liquid. The density wave order parameter decreases, and the shape of the crystal matters less and less. This is connected to the depinning threshold of the WC to electric fields. This connection can be understood as follows (Fukuyama and Lee, 1978, Blatter *et al*, 1994); when the crystal is stiff it is not able to take advantage of all the minima in the disorder potential, and so is not strongly pinned. However, when the crystal becomes soft, it does not cost a lot of energy to deform and take advantage of local minima of the disorder, and the crystal becomes more strongly pinned. Disorder may also have a dominant role in the behavior of the gap near  $\frac{1}{5}$ . Since the crystal is soft, its actual shape is determined by the local disorder. Different shapes give gaps varying by about a factor of two.

The second qualitative result is that the density inhomogeneity in the CF-WC state is quite small in absolute terms (about 20% of the background density) for all  $\nu$ , not just near  $\frac{1}{5}$ . This is at odds with the conventional view that electrons are localized in a WC. The correlation zeroes seem to prefer a more homogeneous state<sup>50</sup>. The electronic WC has to partially melt and become more homogeneous to accommodate Laughlin-Jastrow correlations. This has interesting similarities with earlier ideas concerning cooperative ring-exchanges in a WC, and the melting of the WC (Kivelson, Kallin, Arovas, and Schrieffer, 1986a,b).

## B. Partially polarized Hall Crystal states

Another class of candidates for an inhomogeneous state of CF's stems from the observation of Kukushkin, von Klitzing and Eberl (1999), who measured the spin polarization of various principal fractions as a function of  $E_Z$ , the

<sup>49</sup> Since the CF's see effective flux rather than external flux this point is crucial.

<sup>50</sup> It is easy to see that correlation zeroes become meaningless if electrons are strictly localized; then they do not come close enough for the zeros to be operative.



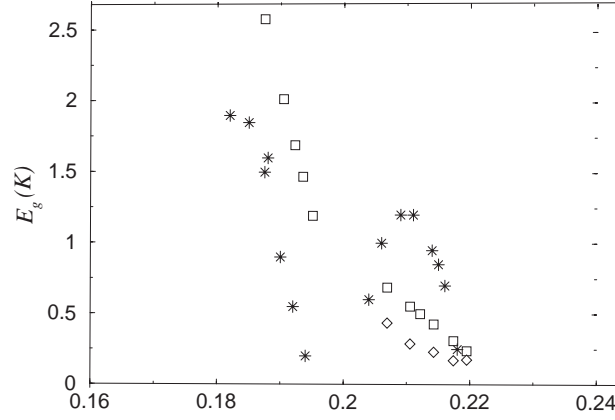


FIG. 15 Experimental (stars, from Jiang *et al* 1991) and theoretical (squares and diamonds, from Narevich, Murthy, and Fertig 2001) gaps versus  $\nu$  for the Wigner crystal. The squares are theoretical gaps for the triangular lattice, while the diamonds represent gaps for an oblique lattice. A change of shape of the lattice just above  $\frac{1}{5}$  could explain the nonmonotonicity of the experimental gap.

Zeeman energy and found polarizations not expected in the CF theory with uniform states. As an example, consider  $\nu = \frac{2}{5}$ . There are two filled CF-LL's, and at small  $E_Z$  we expect that the state is unpolarized with the  $n = 0, \uparrow$  and  $n = 0, \downarrow$  CF-LL's occupied and the rest empty. At large  $E_Z$  we expect a fully polarized state with  $n = 0, \uparrow$  and  $n = 1, \uparrow$  occupied. If these are the only states we are allowed to think about, then at some intermediate  $E_Z$  there should be a first order transition. Such transitions between  $P = 0$  and  $P = 1$  were extensively discussed earlier.

However, Kukushkin, von Klitzing, and Eberl (1999) observe a plateau at half the maximum polarization in a narrow region of  $E_Z$  for  $\frac{2}{5}$ . Analogous plateaus are observed at other principal fractions<sup>51</sup>. Other puzzling phenomena near this transition are that of hysteresis over a range of fillings near  $2/5$  (Cho *et al* 1998) and very slow dynamics (Eom *et al* 2000). One of us (Murthy 2000a) has proposed that these observations might be explained by considering partially polarized Hall Crystals of CFs.<sup>52</sup>

Let us first review some terminology. A state which shows the co-existence of density wave and quantum Hall order (a quantized  $\sigma_{xy}$ ) is called a *Hall Crystal*. Such states were implicit in the ring-exchange theory (Kivelson, Kallin, Arovas, and Schrieffer, 1986a,b), but were first postulated explicitly by Halperin, Tesanovic, and Axel (1986), and analyzed in great detail with explicit examples by Tesanovic, Axel, and Halperin (1989)<sup>53</sup>. A WC is an example of a Hall Crystal, as is the lattice of quasiparticles formed at a filling  $\nu = \frac{1}{3} + \delta$ , provided the density wave is pinned. The reason the partially polarized state at  $2/5$  has to be a Hall Crystal (if it is a crystal at all) is that no changes in the quantized Hall conductance are observed as  $E_Z$  is varied.

Now let us understand the nature of the proposed state (Murthy, 2000a). Imagine that CF's were really free. Then as  $E_Z$  is raised from 0 the system remains in a singlet state until the  $n = 0, \downarrow$  CF-LL (whose energy is increasing) crosses the  $n = 1, \uparrow$  CF-LL (whose energy is decreasing) at a special Zeeman energy  $E_Z^*$ . This is when the system makes the first-order transition. Now let us "turn on" interactions between CF's, which we know are present in the full theory. At  $E_Z^*$  there are twice the number of degenerate states (from the  $n = 0, \downarrow$  and  $n = 1, \uparrow$  CF-LL's) as there are CF's at the Fermi level. The true many-body ground state will therefore be picked by the interactions. This state should be stable in a small range of  $E_Z$  near  $E_Z^*$  determined by the strength of the CF interactions. One possibility is a charge/spin density-wave state.

As we saw in the previous subsection, when the system self-consistently chooses a particular density-wave order with  $p/q$  quanta of effective flux per unit cell, each CF-LL splits up into  $p$  subbands of equal degeneracy (Yoshioka and Fukuyama, 1979). The partially polarized  $\frac{2}{5}$  state has two quanta of effective flux penetrating every unit cell, which means  $p = 2, q = 1$ , and each CF-LL is split into two subbands. The proposed state corresponds to occupying 3 subbands of  $\uparrow$ -CFs and 1 subband of  $\downarrow$ -CFs. Once again a CF-HF calculation allows us to calculate the gap. The

<sup>51</sup> Very recently, polarizations not allowed by translationally invariant CF states have been seen at  $\frac{2}{3}$  by NMR techniques as well: Freytag, Tokunaga, Horvatić, Berthier, Shayegan, and Lévy (2001).

<sup>52</sup> An alternate explanation is provided by Mariani *et al* (2002).

<sup>53</sup> The topological numbers associated with these states had been noticed before by a number of authors: Wannier (1978), Thouless, Kohmoto, Nightingale, and den Nijs (1981), Johnson and Moser (1982), Thouless (1983), MacDonald (1983), Dana, Avron, and Zak (1985), Avron and Seiler (1985), Avron and Yaffe (1986), Kunz (1986). However, Tesanovic, Axel, and Halperin (1989) were the first to give a concrete example where nontrivial topological integers were realized, and to make an important connection between the topological integers and gapless collective excitations. For an extension of the Hall Crystal concept to the FQH regime, see Kol and Read (1993).

theoretical prediction for the range in  $E_Z$  over which the state should be stable is consistent with the experiment (Murthy 2000a).

This possibility can be investigated in more detail in the integer quantum Hall analogue of  $\nu = 2/5$ , which is  $\nu = 2$ . There it can be shown (Murthy 2000b) that there are indeed charge/spin density wave states whose energy (in HF) is less than both the singlet and fully polarized liquid states. It can also be shown that these states are Hall Crystals with a nonzero quantized Hall conductance (Murthy 2000b).

However, there are other possibilities for the partially polarized state at  $\nu = 2/5$ . For example, Apalkov, Chakraborty, Pietiläinen, and Niemelä (2001), have proposed that the CF's belonging to the  $n = 0, \downarrow$  and  $n = 1, \uparrow$  CF-LL's form a Halperin(111) state (Halperin, 1983), which is a liquid. Their proposal is based on a finite-size calculation in a spherical geometry where only the CF's belonging to the two "active" levels were kept, and their interaction was approximated as being a two-body interaction. While their state remains a contender, Apalkov, Chakraborty, Pietiläinen, and Niemelä (2001) wrongly conclude, by neglecting some terms in the ground state energy (see Murthy 2001c), that the ground state energy of the charge/spin density wave state would be higher than the Halperin(111) analogue state.

At the moment it is fair to say that the situation is not resolved. There are some key differences between the two proposals which could be used to distinguish them experimentally. For example, in the liquid state of Apalkov, Chakraborty, Pietiläinen, and Niemelä (2001) one would expect no gapless excitations whatsoever, and no spatial variation in the spin density, and thus the NMR Knight shift. On the other hand, there are generically gapless density excitations in the partially polarized Hall Crystal state (Tesanovic, Axel, and Halperin 1989, Murthy 2000b), and one would expect (at very low temperatures when the crystal is frozen) periodic spatial variation in the Knight shift, with the maximum Knight shift being the same as in the fully polarized state. A complete understanding of these partially polarized states will hinge on future experiments.

## XI. CRITICAL REVIEW OF EHT AND ITS APPROXIMATE SOLUTIONS

The EHT was arrived at by starting with the electronic hamiltonian, introducing a new (pseudovortex) coordinate  $\mathbf{R}_v$ , combining it with the electronic guiding center  $\mathbf{R}_e$  to form the CF variables  $\boldsymbol{\eta}$  and  $\mathbf{R}$ . A (weak) constraint  $\bar{\chi}(\mathbf{q}) \simeq 0$  completed the picture. For  $\nu = 1/2$  the formalism coincides with that of Pasquier and Haldane (1998).

As an exact restatement of the original electronic problem, the EHT is as good as any, with no loss of information, including that pertaining to higher LL's. Its main claim to superiority over other formulations is its amenability to Hartree-Fock and attendant approximations, because the problem is now expressed in terms of CF variables ( $\boldsymbol{\eta}$  and  $\mathbf{R}$ ) which have a natural HF ground state that fills an integral number of LL's.

We discussed two approximation schemes, both of which rely on a nondegenerate HF state, but which differ in how the constraints are handled. Since the separation of the LLL physics (and thus the limit  $m \rightarrow 0$ ) was straightforward in this formalism, we focused on this limit and the LLL.

In a *conserving approximation*, one finds approximate Green's functions that respect the constraints. The time-dependent Hartree-Fock (TDHF) approximation is conserving. It was employed (Murthy 2001a) to show that in the LLL,  $S(q) \simeq q^4$  for gapped states, in compliance with Kohn's theorem. For  $\nu = 1/2$  it was used by Read (1998) to show that the state is compressible, and the fermion effective mass has a logarithmic divergence at the Fermi surface, because the overdamped mode of HLR responsible for all this is generated by summing ladder diagrams. Thus there is no conflict between the HLR results based on CS fermions (that had unit charge) and those based on Composite Fermions with charge  $e^*$  (which vanishes for  $\nu = 1/2$ .) However results that are easily obtained in the CS approach are obtained only following a careful, gauge-invariant calculation in the EHT. It may well be that without the CS results, the compressibility at  $\nu = 1/2$  would have been overlooked in a description that succeeded in displaying dipolar expressions for the electron charge operator. While the primary use of the conserving approximation thus far has been to demonstrate that correct results in the infrared, long-wavelength limit require a proper implementation of constraints, it seems to numerically capture some of the short-distance physics as well, as evidenced by the magnetoexciton dispersions for  $\frac{1}{3}$  and  $\frac{2}{5}$  (Murthy, 2001a, section VIB).

In the other approximation, (*the shortcut*) one uses the preferred charge  $\bar{\rho}^p(\mathbf{q}) = \bar{\rho}(\mathbf{q}) - c^2 \bar{\chi}(\mathbf{q})$  in place of  $\bar{\rho}(\mathbf{q})$  on the grounds that in an exact calculation such a substitution is permitted. This single substitution manages to accomplish at tree level some of the work of the constraints:  $e \rightarrow e^*$ ,  $d \rightarrow d^*$ , and  $S(q) \simeq q^2 \rightarrow S(q) \simeq q^4$ . The constraints are then ignored, so that their effects at higher orders in  $ql$  are not built in, and may or may not be described well by the simple HF calculation that follows. Thus, although the EHT itself is exact, the small parameter  $ql$  is in the shortcut needed because of the way we handled the constraints. Note that it is  $ql$  and not  $ql^*$  that has to be small, as evidenced by the success of the approach near and at  $\nu = 1/2$  where  $l^* \rightarrow \infty$ .

Also missing in the shortcut is the overdamped mode and its attendant consequences. However in problems with a gap or  $T > 0$  this does not seem to matter. On this basis one expects it to work in problems where the potential

favors small  $ql$ , and the problem is still described by CF's<sup>54</sup>.

These expectations were borne out in the computation of gaps, magnetic transitions and  $T$ -dependence of polarization and nuclear spin relaxation, which were compared to results from exact diagonalization, trial wavefunctions and experiment. The very special form the hamiltonian that comes out of this theory does what *ad hoc* hamiltonians parametrized by standard kinetic and interaction terms cannot: Explain a host of phenomena in any given sample (polarization, relaxation) with a single parameter describing electro-electron interactions. We are not aware of any other quantitative, analytical approximation scheme for computing these quantities which is not plagued by singularities as  $m \rightarrow 0$ .

The preferred charge description also answers many qualitative issues: In what sense does an electron bind to its correlation hole (represented here by pseudovortices) to form the CF? How can CF's appear to be free in some magnetic phenomena when they are surely not free? What does the dipole picture really mean? How is an effective mass generated from the interactions alone?

On the down side, the preferred charge as a way to implement constraint is peculiar to the FQHE and unprecedented. We do not know *a priori* how well this procedure will work and how to systematically improve it. We have no internal signal that the CF description is failing as the interaction is varied (say by sending  $\lambda \rightarrow \infty$  in the ZDS interaction) or by mapping  $\nu > 1$  problems into effective LL problems with modified interactions, where Pfaffian states and striped states might be the answer.

## XII. SUMMARY AND OPEN QUESTIONS

The goal of the hamiltonian theory was to start with the hamiltonian for two-dimensional interacting electrons in a magnetic field and arrive at a comprehensive description of fractional quantum Hall phenomena by a sequence of transformations to the final quasiparticles. The final theory was expected to encode and display the wisdom inherited from the study of trial wavefunctions, make the various pictures precise, and permit the computation of all physical quantities (to some accuracy) by the use of standard approximation techniques, such as HF.

Thus we first surveyed the wavefunction approach to see what it could teach us and then described the two hamiltonian approaches: the Chern-Simons (CS) approach and our extension of it, the Extended Hamiltonian Theory (EHT).

The wavefunctions taught us the following. The FQHE ground states are described by incompressible fluids. The wavefunctions for the ground states and quiparticle (or hole) excitations are independent of the mass  $m$  (in the  $m \rightarrow 0$  limit) and is built entirely of LLL wavefunctions. In the Laughlin fractions, the elementary excitation is a vortex of charge  $-1/(2s+1)$  in electronic units. When an extra electron is introduced into the system, it is screened by  $2s$  such vortices, leading to a composite fermion of charge  $e^* = 1/(2s+1)$ . These vortices sit on the electron. In the Jain fractions the wavefunctions are obtained by projection to the LLL. Prior to projection one can see the CF as made of an electron and  $2s$  vortices sitting on them. After projection, many zeros are annihilated, and typically move off the electrons and are no longer organized into vortices. This holds for  $\nu = 1/2$  as well, as is clear when  $\nu = 1/2$  is seen as a limit of nearby Jain fractions. This means that the naive dipole picture based on zeros of the wavefunctions is not tenable. Remarkably, despite the dissolution of vortices on projection, the CF still has the charge of an electron and  $2s$  vortices. The proper interpretation of this requires the hamiltonian approach, in particular the EHT, which in turn is an offshoot of CS theory, which was described next.

*We emphasized that our aim was not simply to get yet another exact reformulation of the problem, but one that lends itself to approximations.* For example the problem in terms of electrons:

$$H = \sum_j \frac{\eta_{ei}^2}{2ml^4} + \frac{1}{2} \sum_{i,j,\mathbf{q}} v(q) e^{i\mathbf{q} \cdot (\mathbf{r}_{ei} - \mathbf{r}_{ej})} \equiv H_0 + V \quad (262)$$

contains all the answers in principle, but is flawed in practice because of the LLL degeneracy of  $H_0$  precludes the use of HF or perturbation theory.

The degeneracy of the noninteracting problem is overcome by the flux attachment transformation of Chern-Simons theory. Lack of space prevented us from discussing the very first application of this idea to the Laughlin fractions by Zhang, Hansson, and Kivelson (1989). They converted the problem to one of composite bosons in zero average field, which led to a detailed analogy to superfluidity. This review focused on CF's, turning to the work of Lopez and Fradkin (1991,1992,1993) who implemented Jain's idea in operator form, by attached  $2s$  flux quanta to electrons

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<sup>54</sup> If one lets the "thickness" parameter of the Zhang-Das Sarma potential become very large, CF's will likely become unstable, and the state will not be describable in the EHT.

converting them to CS fermions described by

$$H_{CS} = \sum_i \frac{(\mathbf{\Pi} + : \mathbf{a}_{cs} :)_i^2}{2m} + V \quad (263)$$

$$\mathbf{\Pi} = \mathbf{p} + e\mathbf{A}^* \quad (\mathbf{A}^* = \frac{\mathbf{A}}{2ps+1}) \quad (264)$$

$$\nabla \times : \mathbf{a}_{cs} : = 4\pi s : \rho : \quad (265)$$

At mean-field one now has a unique vacuum of  $p$ -filled LL's of CF's upon neglecting  $V$  and  $: \mathbf{a}_{cs} :$ . Their effects could be included perturbatively. In particular in an RPA approximation one could include both these and obtain the cyclotron mode at the right frequency, and in the Laughlin fraction, the wavefunction in the long-distance limit (Lopez and Fradkin, 1991,1992,1993).

However, two problems remained: A singular limit as  $m$ , the bare mass of the electron, vanished, and a quasiparticle charge of unity (at tree level) instead of  $e^*$ , owing to the fact that flux tubes and not vortices were attached. These features made it hard to make quantitative predictions.

We mentioned Kalmeyer and Zhang (1992) who considered  $\nu = 1/2$  and pointed out that impurity scattering would be stronger than naively expected since any charge inhomogeneity led to additional gauge flux (by the CS condition) which produced strong scattering.

We then moved to the Halperin, Lee, and Read (1993) treatment of  $\nu \simeq \frac{1}{2}$ , which was extensive and aimed at confronting theory with experiment at a quantitative level. The success of the CF theory in the region of small or zero gap was quite unexpected. They not only highlighted the fact that at  $\nu = 1/2$  the system saw zero gauge field on average and therefore had a Fermi surface, they also argued that the best way to think of the region near  $\nu = 1/2$  was in terms of CF's seeing an effective field  $B^* = B - B_{1/2}$ . This meant the particles would bend with a radius  $R^* = \frac{\hbar k_F}{eB^*}$  with  $k_F = \sqrt{4\pi n}$ , a result that was verified experimentally by Kang *et al* (1993), Goldman, Su, and Jain (1994), and Smet *et al* (1996).

HLR computed the electromagnetic response within the RPA. They showed that the system was as compressible as a traditional Fermi liquid. It had a longitudinal conductivity  $\sigma_{xx} = \frac{e^2}{8\pi} \frac{q}{k_F}$ . This was tested by the damping and velocity shift of surface acoustic waves by Willett *et al* (1990,1996) who also found that away from  $\nu = 1/2$ , there was a resonance in the velocity shift when the wavelength of the surface acoustic wave coincided with  $2R^*$ .

HLR identified an overdamped mode in the density-density correlation with dispersion  $\omega \simeq iq^3 v(q)$ . This mode, responsible for the compressibility of the system, also produces a (logarithmic) divergence in the CF mass by entering its self-energy. This in turn leads to a gap  $E \simeq 1/(p \ln p)$  for  $\nu = p/(2p+1)$  as  $p \rightarrow \infty$ . It has been shown that the mass divergences do not affect bosonic (e.g., density-density) correlations<sup>55</sup>

These successes notwithstanding, the HLR approach had some room for improvement. It did not allow a clear separation of the LLL physics, i.e., it did not have a smooth  $m \rightarrow 0$  limit. The CS fermion, obtained by attaching two flux tubes to electrons had a charge of unity (and not  $e^* = 0$ ). It did not make contact with the dipole picture (Read, 1994,1996) based on a wavefunction analysis.

The Extended Hamiltonian Theory (EHT) was devised by us to address some of these issues. *The key idea is that in order to discuss the correlation hole that accompanies the electron to form the CF in tractable form, one must enlarge the Hilbert space to describe collective charge degrees of freedom and to place a suitable number of constraints.* This idea, along with expressions for the new coordinates, and a certain change of variables, all arose in our original tortuous route (Shankar and Murthy, 1997, Murthy and Shankar, 1998a) from electrons to CF's. In this review, we spared the reader details of the historical route and gave an axiomatic description (Murthy and Shankar, 2002) of the EHT.

Here we begin with the electronic hamiltonian Eqn. (262), and add for each electron a new pseudovortex guiding center coordinate  $\mathbf{R}_v$  (whose algebra corresponding to charge  $-c^2$ ). Thus the Hilbert space is bigger now and  $\mathbf{R}_v$  has no dynamics.

If we are interested in physics at the cyclotron scale, we can focus on the kinetic term. The mass  $m$  appears here as it should, in  $\omega_0$ . The collective coordinate formed out of  $\boldsymbol{\eta}_e$  carries the entire Hall current. (This answers the question of who carries the Hall current at  $\nu = \frac{1}{2}$  when  $e^* = 0$ .) In fact they carry the Hall current even at gapped fractions where  $e^* \neq 0$ , thus substantiating the general belief that the Hall current is not affected by disorder because it is carried by collective coordinates. Having identified them as our oscillators, (for which we have a specific hamiltonian) we have paved the way for a detailed analysis of the Hall response. We also showed how extract an effective LLL theory by approximately integrating out the coordinate  $\boldsymbol{\eta}_e$  in our discussion of LL mixing.

<sup>55</sup> Kim, Furusaki, Wen, and Lee (1994), Kim, Lee, Wen, and Stamp (1994), Kim, Lee, and Wen (1995), Stern and Halperin (1995).

Let us now proceed to drop higher LL's and work with

$$\bar{H}(\bar{\rho}) = V = \frac{1}{2} \sum \bar{\rho}(q) v(q) e^{-(ql)^2/2} \bar{\rho}(q) \quad (266)$$

$$\bar{\rho}(\mathbf{q}) = \sum_j \exp \left[ -i\mathbf{q} \cdot \left( \mathbf{r}_j - \frac{\mathbf{z} \times \mathbf{\Pi}_j}{1+c} \right) \right] \equiv \sum_j \exp [-i\mathbf{q} \cdot \mathbf{R}_{ej}] \quad (267)$$

The densities  $\bar{\rho}(\mathbf{q})$  obey the magnetic translation algebra since  $\mathbf{R}_e$  is the electron guiding center coordinate *in the CF basis*. The advantage of this basis is that the velocity operator  $\mathbf{\Pi}$  sees a weaker field  $B^*$ , leading to a nondegenerate HF ground state.

The hamiltonian commutes with the operators

$$\bar{\chi}(\mathbf{q}) = \sum_j \exp \left[ -i\mathbf{q} \cdot \left( \mathbf{r}_j + \frac{\mathbf{z} \times \mathbf{\Pi}_j}{c(1+c)} \right) \right] \equiv \sum_j \exp [-i\mathbf{q} \cdot \mathbf{R}_{vj}]. \quad (268)$$

The pseudovortex densities  $\bar{\chi}(\mathbf{q})$  thus form a closed algebra, the symmetry algebra of  $H$ .

In the first derivation of the theory (Shankar, 1999), the following constraint naturally emerged:

$$\bar{\chi}(\mathbf{q})|\text{physical state}\rangle = 0. \quad (269)$$

In the extended approach (Murthy and Shankar, 2002),  $\mathbf{R}_v$  is a cyclic coordinate with no dynamics. One is free to supplement the theory with the above constraint since nothing physical depends on the auxiliary coordinate  $\mathbf{R}_v$ . We made this choice so that there would be just one set of equations to deal with.

We described two ways to proceed, the choice being dictated by what we want to calculate; the conserving approximation and the shortcut that used the preferred charge.

For situations where the symmetries of  $H$  are important one uses the conserving approximation, i.e., one in which the constraint is respected at the level of Greens functions. This amounts to ensuring gauge invariance. We reviewed the compressibility paradox where gauge invariance made all the difference. The paradox concerned the system at  $\nu = \frac{1}{2}$  which Halperin and Stern (1998) argued must be compressible (as had been predicted by HLR). This result seemed to be at odds with our operator description in which the charge was dipolar. Halperin and Stern gave heuristic arguments for how a system of dipoles could still be compressible if their hamiltonian had  $K$ -invariance- invariance of the energy under the shift all momenta, a symmetry first noted by Haldane, and which appeared in our work as part of a gauge symmetry. The more detailed analysis of Stern, Halperin, von Oppen, and Simon (1999) drove the point home. They considered first a model where the flux tubes were spread over a distance  $1/Q$  and the hamiltonian had  $K$ -invariance to sufficiently high order in  $Q$  to examine the question of compressibility. The density-density correlation (now of the dipole-dipole form) was computed in the RPA (which was exact in the limit  $Q \rightarrow 0$ ) and the factors of  $q^2$  in the numerators from the dipoles were cancelled by the exchange of the overdamped mode. They then showed that in the actual FQHE problem,  $K$ -invariance could be built in if the Landau parameter  $F_1 = -1$ . The conserving approximation was also employed by Read (1998) within the Pasquier-Haldane (1998) formalism for  $\nu = 1$  bosons (to which our theory reduces if we set  $c = 1$  or  $\nu = \frac{1}{2}$ ). Read (1998) established that the fermions interacted with each other as dipoles by exchanging a transverse collective mode, but that as  $q \rightarrow 0$ , the propagator for this mode produced enough negative powers of  $q$  to overturn the  $q^2$  coming from the dipolar factors from the two ends. These calculations are important in showing that if constraints are taken into account, the correct charge of the CF will emerge and that despite the weaknesses in the wavefunction based arguments leading to the dipole picture (see Section I.D.3), the predicted dipole moment is correct in the above sense. Murthy (2001a) was able to show that in the gapped fractions one could obtain structure factors in accord with Kohn's (1961) theorem ( $\bar{S} \simeq q^4$ ) in a conserving calculation. He also showed how to compute magnetoexciton dispersions.

In all other situations (where a gap or temperature or both suppress the deep infrared region  $\omega \simeq q^3 v(q)$ ) the shortcut using  $\bar{\rho}^p(\mathbf{q})$  was the weapon of choice. Here we make the replacement  $\bar{\rho} \rightarrow \bar{\rho}^p = \bar{\rho} - c^2 \bar{\chi}$ , which is allowed in the exact theory. This choice allows one to employ naive Hartree-Fock calculations that respect Kohn's Theorem ( $\bar{S} \simeq q^4$ ). As a bonus the CF charge and dipole moment emerge in the power series expansion of  $\bar{\rho}^p$ . It is only in this sense that the CF can be viewed as the union of an electron and a correlation hole of charge  $-c^2$ . With these features built in at tree level, the usual approximations such as Hartree-Fock are applicable as long as the large  $ql$  region is avoided by the potential.

The EHT gives a uniform and precise description of the internal structure of the CF. Whereas in the wavefunction based description, the Laughlin fractions allowed for a simple picture of the CF (an electron bound to  $2s$  vortices), and the rest of the Jain series (upon projection) did not, in the EHT the CF is viewed as an electron plus a pseudovortex. Especially interesting is the case of  $\nu = \frac{1}{2}$ . The expansion for  $\bar{\rho}^p$  in a power series shows that it begins with a term that couples to an external electric field exactly as a dipole moment of strength  $l^2 \mathbf{z} \times \mathbf{p}$  would. This formula, in operator form, makes no reference to zeros of the wavefunction or vortices, neither of which is robust. It does not

have the problems of the wavefunction-based dipole picture. These problems arose upon antisymmetrization, which we carry out by expressing  $\bar{\rho}^p$  in second quantized form in terms of fermionic operators. We saw that the place look for dipoles is not in the wavefunction, but in correlation functions at high frequencies at and above the CF Fermi energy.

The operator approach gives a concrete realization of one of the primary expectations in FQHE: Once the kinetic energy is quenched by restricting electrons to the LLL, it will be resurrected by interactions and that this low-energy problem will be characterized by one common scale, the electron-electron interaction. Eqn. (61) which gives  $H$  as a quadratic function of  $\bar{\rho}$ , (or  $\bar{\rho}^p$  if we use the preferred combination) embodies all these expectations.

The HF approximation to  $H(\bar{\rho}^p)$  was used to compute transport gaps (Murthy and Shankar, 1999, Shankar, 2001) good to within 10 – 20% for potentials that vanish rapidly with  $ql$ . It could be used to explore magnetic transitions from one quantized value of magnetization to the next (Shankar, 2000, 2001). In the absence of disorder it is clear that all physical quantities pertaining to the FQHE (restricted to the LLL) are functionals of the potential  $v(q)$ . For the Zhang-Das Sarma potential which we use for most of our work, this means a function of  $\lambda$ . We saw that given a value of  $\lambda$  from experiment, we did not need several masses for several phenomena, they all came from one potential (Shankar, 2000, 2001). This is because the hamiltonian for the CF's is rather unusual<sup>56</sup>, and contains the kinetic and potential terms in a monolithic form. Hidden in it are the various mass scales  $m_a$  and  $m_p$  appropriate to various phenomena (activation, polarization), all functionals of the interaction  $v(q)$ .

The operator approach also clarified the question of whether or not CF's are free. Given the prominent variations in the magnetoexciton dispersions and the fact that it takes two very different masses to describe polarization and activation, it is clear that they are not. But why do they appear to be free for some magnetic phenomena? Our theory (Shankar, 2000, 2001) shows that it is an accident coming from rotational invariance and  $d = 2$ .

The EHT allows us to compute physical quantities at  $T > 0$ , such as polarization and relaxation rates for gapless states (Shankar, 2000, 2001). In the experiments of Dementyev *et al* (1999), a  $\lambda$  determined from one polarization data point gives the polarization and relaxation curves for two tilts and a range of temperatures. This is to be contrasted with attempts to fit the data with a mass and interaction pair  $(m, J)$ , where the four curves require four nonoverlapping values of these pairs. In the case of Melinte *et al* (2000), Freytag (2001), the  $1/T_1$  predictions (that vary over orders of magnitude) are off by a factor of 2, but the polarization data are well described by a single  $\lambda$ . Once again the success can be traced back to the fact that the CF hamiltonian we use is of a nonstandard form parametrized by a  $\lambda$ , which in turn determines all the mass and energy scales relevant to each given process.

When calculating the polarization of gapped states, it turns out to be essential to take into account spin-waves for spontaneously polarized cases. This is done (Murthy, 2000c) by mapping the low-energy dynamics of the problem on to the continuum quantum ferromagnet treated in the large- $N$  approximation (Read and Sachdev, 1996), the parameters of which are extracted from the HF treatment of the problem in the hamiltonian theory. The results (Murthy, 2000c) agree extremely well with experiments (Khandelwal *et al*, 1998, Melinte *et al*, 2000) up to very high temperatures. This calculation, like all the others in the Hamiltonian theory, is in the thermodynamic limit and free from finite-size effects.

The EHT allows us to calculate the gaps of the Wigner crystal in terms of CF's (Narevich, Murthy, and Fertig, 2001). These gaps are off by a factor of 2 when compared to experiment (Jiang *et al*, 1990, Jiang *et al*, 1991). This should be contrasted to previous approaches in which the gaps were off by two orders of magnitude. This approach also allows us to consider inhomogeneous states (Murthy, 2000a) with polarizations not allowed by the CF theory of homogeneous ground states.

Hopefully we have succeeded in establishing that the hamiltonian theory of the FQHE is a comprehensive scheme for addressing and answering a variety of qualitative and quantitative questions, for gapped and gapless states, at zero and nonzero temperatures.

We have given the reader a taste of what can be done using this formalism. While many things have been clarified, there are many open problems where we believe this approach may be fruitfully applied.

One outstanding open problem is that of computing transport coefficients in the quantum Hall regime from a microscopic theory. By identifying the collective mode which carries the Hall current, we have set the stage for a study of transport in the presence of disorder.

The FQH edge<sup>57</sup> is another open problem where the hamiltonian approach is applicable. After Wen's (1990a,b, 1992) description of the edge as a chiral Luttinger liquid, other descriptions have appeared based on wavefunction and field-theoretic approaches<sup>58</sup>. The field-theoretic descriptions are all effective theories whose connection to the electron problem has not been rigorously established. While the wavefunction approaches are microscopic, it is impractical to calculate time-dependent response functions in them. As we have seen in the extended formalism, we have an exact rewriting of the microscopic electron problem, but with the added advantage of a nondegenerate starting point.

<sup>56</sup> Consider for example Eqn. (154) for  $\nu = \frac{1}{2}$ .

<sup>57</sup> For an excellent review, see Kane and Fisher (1997).

<sup>58</sup> For wavefunction-based approaches, see: Zülicke, U., and A.H.MacDonald (1999), Goldman and Tsiper (2001), Mandal and Jain (2001b). For field-theoretic descriptions, see: Lee and Wen (1998), Lopez and Fradkin (1999), Levitov, Shytov, and Halperin (2001).

The calculation of edge reconstructions (MacDonald, Yang and Johnson, 1983, Chamon and Wen, 1994) in CF-HF and the number and dispersions of the edge collective modes in TDHF seem to be very accessible in the hamiltonian approach. One of the interesting things to consider in the edge problem is tunneling<sup>59</sup>. For this one needs a description of electron creation and destruction operators within the CF basis, which is an open problem.

Since there is a gap, the effects of disorder on the Jain series ought to be describable in some simple approximation, such as the self-consistent Born approximation for single-particle properties. Such a treatment seems to capture many of the experimental facts at  $\nu = 1$  (see Murthy, 2001b), such as the reduction of the transport gap due to disorder, the variation of the transport gap as a function of  $E_Z$  (see, for example, Schmeller *et al* (1995)), the polarization at  $\nu = 1$ <sup>60</sup>, etc. Such an approach ought to be applicable to the gapped fractions. The effect of disorder on excitons is also interesting, and can be treated in a "self-consistent exciton approximation" (Kallin and Halperin 1985).

A more ambitious problem that might benefit from the hamiltonian approach is the question of whether the  $\nu = \frac{1}{2}$  state remains metallic at  $T = 0$ , when disorder is included. The CS formalism of HLR (with its logarithmic corrections) suggests that metallic behavior disappears. It is known that noninteracting fermions are always localized in two dimensions, regardless of how weak disorder is (Abrahams, Anderson, Licciardello, and Ramakrishnan, 1979). By extension, it seems plausible that a Fermi liquid, which is adiabatically connected to noninteracting fermions, should also be an insulator on the longest length scales. The  $\nu = \frac{1}{2}$  system is rather unique, in that the state is produced by interactions. The Extended hamiltonian has additional symmetries absent in the zero-field problem.

We have excluded from this review many interesting topics to which the hamiltonian is applicable, such as double-layer systems<sup>61</sup>, paired states<sup>62</sup>, etc. These omissions reflect the double constraints of limited space and our own lack of expertise.

## Acknowledgments

Over the years, as we have tried to understand the FQHE, we have acquired valuable insights from countless discussions with many of our colleagues and gratefully acknowledged them in our various papers. To mention them all here is unrealistic. We take this opportunity to thank them all collectively, while apologizing for the anonymity. This work, performed over a period of six years was made possible from grants DMR-0071611 (GM), and DMR- 0103639 (RS) from the National Science Foundation. Without this support, it would not have been possible for us to pursue this fascinating subject to our heart's content.

## APPENDIX A: Matrix elements

Many of the calculations performed in this paper deal with the preferred density  $\bar{\rho}^p$ . In second quantization we write it as

$$\bar{\rho}^p(\mathbf{q}) = \sum_{m_2 n_2; m_1 n_1} d_{m_2 n_2}^\dagger d_{m_1 n_1} \rho_{m_2 n_2; m_1 n_1} \quad (\text{A1})$$

where  $d_{m_2 n_2}^\dagger$  creates a particle in the state  $|m_2 n_2\rangle$  where  $m$  is the angular momentum and  $n$  is the LL index. They are related to the CF cyclotron and guiding center coordinates,  $\mathbf{R}$  and  $\boldsymbol{\eta}$  as follows. Let

$$b = \frac{R_x - iR_y}{\sqrt{2}l^{*2}} \quad b^\dagger = \frac{R_x + iR_y}{\sqrt{2}l^{*2}} \quad (\text{A2})$$

where  $l^* = l/\sqrt{1 - c^2}$  is the CF magnetic length. These obey the oscillator algebra

$$[b, b^\dagger] = 1 \quad (\text{A3})$$

<sup>59</sup> The theory was described in a series of beautiful papers by Kane and Fisher (1992a,b, 1994). Recently there have appeared some truly remarkable tunneling experiments where the data span several orders of magnitude of voltage: Grayson *et al* (1998), Chang *et al* (2001)

<sup>60</sup> Tycko *et al* (1995), Barrett *et al* (1995), Aifer *et al* (1996), Manfra *et al* (1996)

<sup>61</sup> For some of the early theoretical references see: Fertig (1989), Wen and Zee (1992, 1993), Ezawa and Iwazaki (1993), Yang *et al* (1994), Moon *et al* (1995). For a recent review, see Girvin and MacDonald (1997): For recent experiments in double-layer systems see: Spielman *et al* (2001), Kellogg *et al* (2002)

<sup>62</sup> The experiment that stimulated this subfield was the observation of a plateau in the Hall resistance at  $\nu = \frac{5}{2}$ : Willett, Eisenstein, Störmer, Tsui, Gossard, and English (1987). For some of the theoretical references, see: Haldane and Rezayi (1988), Moore and Read (1991), Greiter, Wen, and Wilczek (1992), Ho (1995), Park, Melik-Alaverdian, Bonesteel, and Jain (1998), Morf (1998), Bonesteel (1999), Read and Green (2000).

given

$$[R_x, R_y] = -il^{*2}. \quad (\text{A4})$$

Similarly we define, in terms of the cyclotron coordinates,

$$a = \frac{\eta_x + i\eta_y}{\sqrt{2l^{*2}}} \quad a^\dagger = \frac{\eta_x - i\eta_y}{\sqrt{2l^{*2}}} \quad (\text{A5})$$

which obey the oscillator algebra

$$[a, a^\dagger] = 1 \quad (\text{A6})$$

given

$$[\eta_x, \eta_y] = il^{*2}. \quad (\text{A7})$$

The states  $|mn\rangle$  are just the tensor products

$$|mn\rangle = \frac{(b^\dagger)^m}{\sqrt{m!}} \frac{(a^\dagger)^n}{\sqrt{n!}} |00\rangle \quad (\text{A8})$$

where  $|00\rangle$  is annihilated by both  $a$  and  $b$ .

We will now show that

$$\begin{aligned} \langle m_2 | e^{-i\mathbf{q}\cdot\mathbf{R}} | m_1 \rangle &= \sqrt{\frac{m_2!}{m_1!}} e^{-x/2} \left( \frac{-iq + l^*}{\sqrt{2}} \right)^{m_1 - m_2} \\ &\times L_{m_2}^{m_1 - m_2}(x) \end{aligned} \quad (\text{A9})$$

where

$$x = q^2 l^{*2} / 2, \quad q_\pm = q_x \pm iq_y \quad (\text{A10})$$

$L$  is the associated Laguerre polynomial, and  $m_1 \geq m_2$ . If  $m_1 < m_2$  one may invoke the relation

$$\langle m_2 | e^{-i\mathbf{q}\cdot\mathbf{R}} | m_1 \rangle = \langle m_1 | e^{+i\mathbf{q}\cdot\mathbf{R}} | m_2 \rangle^*. \quad (\text{A11})$$

Likewise to establish Eqn. (A9), consider the coherent states

$$|z\rangle = e^{b^\dagger z} |0\rangle = \sum_{m=0}^{\infty} \frac{|m\rangle}{\sqrt{m!}} z^m \quad (\text{A12})$$

with the inner product

$$\langle \bar{z} | z \rangle = e^{\bar{z}z} \quad (\text{A13})$$

First we write from the definitions given above

$$\begin{aligned} \langle \bar{z} | e^{-i\mathbf{q}\cdot\mathbf{R}} | z \rangle &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{\bar{z}^{m_2}}{\sqrt{m_2!}} \frac{z^{m_1}}{\sqrt{m_1!}} \langle m_2 | e^{-i\mathbf{q}\cdot\mathbf{R}} | m_1 \rangle \end{aligned} \quad (\text{A14})$$

$$\equiv R(\bar{z}, z, \mathbf{q}). \quad (\text{A15})$$

On the other hand

$$\langle \bar{z} | e^{-i\mathbf{q}\cdot\mathbf{R}} | z \rangle = \langle \bar{z} | \exp\left(-\frac{il^*}{\sqrt{2}}(q_+ b^\dagger + q_- b)\right) | z \rangle \quad (\text{A16})$$

$$= \left\langle \bar{z} - \frac{il^*}{\sqrt{2}}q_+ \middle| z - \frac{il^*}{\sqrt{2}}q_- \right\rangle e^{q^2 l^{*2} / 4} \quad (\text{A17})$$

$$= \exp\left[\bar{z}z - \frac{il^*}{\sqrt{2}}(\bar{z}q_- + q_+ z)\right] e^{-q^2 l^{*2} / 4} \quad (\text{A18})$$

$$\equiv R(\bar{z}, z, \mathbf{q}) \quad (\text{A19})$$



Comparing Eqns. (A14-A18) and matching powers of  $\bar{z}^a z^b$  we obtain Eqn. (A9) if we recall

$$L_{m_2}^{m_1-m_2}(x) = \sum_{t=0}^{m_2} \frac{m_1!}{(m_2-t)!(m_1-m_2+t)!} \frac{(-1)^t}{t!} x^t \quad (\text{A20})$$

Likewise, to establish

$$\langle n_2 | e^{-i\mathbf{q} \cdot \boldsymbol{\eta}} | n_1 \rangle = \sqrt{\frac{n_2!}{n_1!}} e^{-x/2} \left( \frac{-iq_- l^*}{\sqrt{2}} \right)^{n_1-n_2} L_{n_2}^{n_1-n_2}(x) \quad (\text{A21})$$

(again for  $n_1 \geq n_2$ ) we just need to remember that the commutation rules of the components of  $\boldsymbol{\eta}$  have a minus sign relative to those of  $\mathbf{R}$ , which exchanges the roles of creation and destruction operators and hence  $q_+$  and  $q_-$ .

Now we consider matrix elements of  $\bar{\rho}, \bar{\chi}, \bar{\rho}^p$ . As a first step, let us express the operators  $\mathbf{R}_e$  and  $\mathbf{R}_v$  in terms of CF guiding center and vortex coordinates  $\mathbf{R}$  and  $\boldsymbol{\eta}$ . We have seen that in the CF representation

$$\mathbf{R}_e = \mathbf{r} - l^2 \frac{\hat{\mathbf{z}} \times \boldsymbol{\Pi}}{1+c} = \mathbf{R} + \boldsymbol{\eta} c \quad (\text{A22})$$

if we recall  $l^2 = l^{*2}(1-c^2)$ .

It can similarly be shown that

$$\mathbf{R}_v = \mathbf{r} + l^2 \frac{\hat{\mathbf{z}} \times \boldsymbol{\Pi}}{c(1+c)} = \mathbf{R} + \boldsymbol{\eta}/c. \quad (\text{A23})$$

Thus in first quantization

$$\bar{\rho}^p = \bar{\rho} - c^2 \bar{\chi} \quad (\text{A24})$$

$$\bar{\rho} = \sum_i \exp(-i\mathbf{q} \cdot \mathbf{R}_i) \exp(-i\mathbf{q} \cdot \boldsymbol{\eta}_i c) \quad (\text{A25})$$

$$\bar{\chi} = \sum_i \exp(-i\mathbf{q} \cdot \mathbf{R}_i) \exp(-i\mathbf{q} \cdot \boldsymbol{\eta}_i / c) \quad (\text{A26})$$

$$(\text{A27})$$

Armed with Eqns. (A9 and A21) we may finally write for the matrix elements of  $\bar{\rho}(\mathbf{q})$  defined in Eqn. (A1),

$$\begin{aligned} \bar{\rho}_{m_2 n_2; m_1 n_1} &= \\ &\sqrt{\frac{m_2!}{m_1!}} e^{-x/2} \left( \frac{-iq_+ l^*}{\sqrt{2}} \right)^{m_1-m_2} L_{m_2}^{m_1-m_2}(x) \\ &\otimes \left[ \sqrt{\frac{n_2!}{n_1!}} \left( \frac{-icq_- l^*}{\sqrt{2}} \right)^{n_1-n_2} e^{-xc^2/2} L_{n_2}^{n_1-n_2}(xc^2) \right. \\ &\quad \left. - c^2 \cdot f \cdot \left( \frac{-iq_- l^*}{\sqrt{2}c} \right)^{n_1-n_2} e^{-x/2c^2} L_{n_2}^{n_1-n_2}(x/c^2) \right] \\ &\equiv \rho_{m_2 m_1}^m \otimes \rho_{n_2 n_1}^n \end{aligned}$$

Superscripts on  $\rho_{m_2 m_1}^m$  and  $\rho_{n_2 n_1}^n$  which will be apparent from the subscripts, will usually be suppressed.

## APPENDIX B: Hall response in the extended picture

To compute the DC Hall conductance we just need the zero momentum component of the current operator

$$\mathbf{J}(0) = \frac{\partial H}{\partial \mathbf{A}_{\mathbf{q}=0}} = \frac{e}{m} \sum_j \boldsymbol{\Pi}_j = -\frac{e}{ml^2} \sum_j \hat{\mathbf{z}} \times \boldsymbol{\eta}_{ej}. \quad (\text{B1})$$

Note that the current at  $q = 0$  depends only on the cyclotron coordinate, just as it depended only on the oscillator coordinate in the small treatment. Upon coupling the system to an external potential  $\Phi(\mathbf{q})$ ,  $H$  becomes

$$H = \sum_j \frac{\eta_{ej}^2}{2ml^4} + e\Phi(\mathbf{q}) \sum_j e^{-i\mathbf{q} \cdot \mathbf{r}_j} \quad (\text{B2})$$

If we now keep the interaction to  $\mathcal{O}(\mathbf{q})$ , and recall  $\mathbf{r}_e = \mathbf{R}_e + \boldsymbol{\eta}_e$ , we obtain

$$H = \sum_j \frac{\eta_{ej}^2}{2ml^4} + e\Phi(\mathbf{q}) \sum_j (-i\mathbf{q} \cdot (\mathbf{R}_{ej} + \boldsymbol{\eta}_{ej})) + \dots \quad (\text{B3})$$

If we complete squares on  $\boldsymbol{\eta}_{ej}$ , we can read off its mean value in the presence of  $\Phi$ , or the corresponding electric field  $i\mathbf{q}\Phi$ . From this we obtain a mean current  $\mathbf{J}(0)$  corresponding to the right Hall conductance of  $ne/B$ .

The exercise should make it clear that the oscillator coordinates are just the collective coordinates formed from  $\boldsymbol{\eta}_e$  at small  $q$ . If we go to higher orders in  $q$  we will find that the current involves both  $\boldsymbol{\eta}_e$  and  $\mathbf{R}_e$ . Our extended formalism allows one to explore corrections due to this mixing in a small- $q$  expansion.

Note that in this approach the magnetic moment of Simon *et al* need not be put in by hand since  $\boldsymbol{\eta} + e$  is still; in the picture and can respond to a slowly varying magnetic field by a changing zero-point energy.

The objections of Lee *et al* (1999) on the CF Hall conductance are moot since we do not add resistivities as in Eqn. (107) but rather conductivities (of the  $\boldsymbol{\eta}_e$  and CF variables).

### APPENDIX C: Proof of Hartree-Fock nature of trial states

Consider

$$\langle f|H|i\rangle = \langle \mathbf{p}|d_f H d_i^\dagger|\mathbf{p}\rangle \quad (\text{C1})$$

where  $|\mathbf{p}\rangle$  stands for the (ground) state with  $p$ -filled LL, and  $i, f$  label single-particle excitations on top of this ground state. We want to show that this matrix element vanishes if  $i \neq f$ , i.e., the hamiltonian does not mix these putative HF particle states. (This result was established for the small  $q$  theory by Murthy, 1999). The proof, which relies on just the rotational invariance of the potential, applies with trivial modifications to the hole states, i.e., to

$$\langle \mathbf{p}|d_f^\dagger H d_i|\mathbf{p}\rangle. \quad (\text{C2})$$

This matrix element in question takes the schematic form

$$\langle f|H|i\rangle = \int_q \langle \mathbf{p}|d_f d_1^\dagger d_2 d_3^\dagger d_4 d_i^\dagger|\mathbf{p}\rangle \rho_{12}(\mathbf{q}) \rho_{34}(-\mathbf{q}) \quad (\text{C3})$$

where 1 stands for  $m_1 n_1$  and so on, and  $\int_q$  stands for an integral over a *rotationally invariant measure*:

$$\int_q = \frac{1}{2} \int \frac{d^2 q}{4\pi^2} v(q) e^{-q^2 l^2/2}. \quad (\text{C4})$$

Now we use Wick's theorem and perform pairwise contractions on the vacuum expectation value, bearing mind that

- We cannot contract the indices 1 and 2 or 3 and 4 since this will require that  $q = 0$  at which point the measure (which contains the potential) vanishes.
- If we contract  $i$  and  $f$  we already have the desired result.

Here is a representative of the contractions we can get:

$$\rho_{12} \rho_{34} \delta_{f1} (1 - n_1^F) \delta_{23} (1 - n_2^F) \delta_{4i} (1 - n_4^F). \quad (\text{C5})$$

where  $n_1^F$  is the Fermi function for the LL labeled by  $n_1$

$$n_1^F = \theta(p - 1 - n_1) \quad (\text{C6})$$

and so on. Since  $f = 1$  and  $i = 4$ , the factor  $(1 - n_2^F)(1 - n_4^F) = 1$ . The integrand assumes the form

$$\begin{aligned} \sum_{m_2=0}^{\infty} \sum_{n_2=p}^{\infty} \rho_{f2}(\mathbf{q}) \rho_{2i}(-\mathbf{q}) &= \\ &= \left[ \sum_{m_2=0}^{\infty} \rho_{m_f m_2}(\mathbf{q}) \rho_{m_2 m_i}(-\mathbf{q}) \right] \left[ \sum_{n_2=p}^{\infty} \rho_{n_f n_2}(\mathbf{q}) \rho_{n_2 n_i}(-\mathbf{q}) \right] \\ &= \delta_{m_f m_i} \sum_{n_2} q_-^{n_i - n_f} F(|q|) \end{aligned}$$

where we have also used the fact that  $e^{-i\mathbf{q} \cdot \mathbf{R}} \cdot e^{i\mathbf{q} \cdot \mathbf{R}} = I$  (the identity operator) in doing the sum over  $m_2$ , and  $F(|q|)$  is some rotationally invariant function. It follows that every term in the sum over  $n_2$  vanishes unless  $n_f = n_i$  due to the angular integral in  $\mathbf{q}$ .

#### APPENDIX D: On the conserving nature of TDHF

We will now verify that the constraint is a left-eigenvector of  $\mathcal{H}$ , as required of the conserving approximation. To this end we will need the matrix elements

$$\tilde{\rho}_{n_1 n_2}(\mathbf{q}) = \langle n_1 | e^{-ic\mathbf{q} \cdot \boldsymbol{\eta}} | n_2 \rangle \quad (\text{D1})$$

$$\tilde{\chi}_{n_1 n_2}(\mathbf{q}) = \langle n_1 | e^{-\frac{i}{c}\mathbf{q} \cdot \boldsymbol{\eta}} | n_2 \rangle \quad (\text{D2})$$

Note that only the cyclotron part of the electron and pseudovortex coordinates appear in these exponentials. An important result we will need is

$$\sum_n \tilde{\chi}_{n_1 n}(\mathbf{q}_1) \tilde{\rho}_{nn_2}(\mathbf{q}_2) = \langle n_1 | e^{-\frac{i}{c}\mathbf{q}_1 \cdot \boldsymbol{\eta}} e^{-ic\mathbf{q}_2 \cdot \boldsymbol{\eta}} | n_2 \rangle \quad (\text{D3})$$

$$= e^{-\frac{i}{2}\mathbf{Q}_1 \times \mathbf{Q}_2} \langle n_1 | e^{-i(\frac{\mathbf{q}_1}{c} + c\mathbf{q}_2) \cdot \boldsymbol{\eta}} | n_2 \rangle \quad (\text{D4})$$

$$\mathbf{Q} = \mathbf{q}^{l*} \quad (\text{D5})$$

where we have used the completeness of the states  $|n\rangle$  and the commutation rules obeyed by  $\boldsymbol{\eta}$ . Finally we separate the exponentials in the reverse order to get the second useful identity

$$\begin{aligned} \sum_n \tilde{\chi}_{n_1 n}(\mathbf{q}_1) \tilde{\rho}_{nn_2}(\mathbf{q}_2) \\ = e^{-i\mathbf{Q}_1 \times \mathbf{Q}_2} \sum_n \tilde{\rho}_{n_1 n}(\mathbf{q}_2) \tilde{\chi}_{nn_2}(\mathbf{q}_1) \end{aligned} \quad (\text{D6})$$

Note that  $\tilde{\rho}$  and  $\tilde{\chi}$  do not commute, even though  $\bar{\rho}$  and  $\bar{\chi}$  do.

Let us now right-multiply the putative left-eigenvector  $\tilde{\chi}$  by  $\mathcal{H}$

$$\begin{aligned} \sum_{n'_1 n'_2} \tilde{\chi}_{n'_1 n'_2}(\mathbf{q}) \mathcal{H}(n'_1 n'_2; n_1 n_2; \mathbf{q}) = \\ (\epsilon(n_1) - \epsilon(n_2)) \tilde{\chi}_{n_1 n_2}(\mathbf{q}) \\ + \sum_{n'_1 n'_2} (N_F(n'_2) - N_F(n'_1)) \frac{v(q)}{2\pi(l^*)^2} e^{-q^2 l^2 / 2} \tilde{\chi}_{n'_1 n'_2}(\mathbf{q}) \tilde{\rho}_{n'_2 n'_1}(-\mathbf{q}) \tilde{\rho}_{n_1 n_2}(\mathbf{q}) \\ - \sum_{n'_1 n'_2} (N_F(n'_2) - N_F(n'_1)) \int \frac{d^2 s}{(2\pi)^2} v(s) e^{-\frac{s^2 l^2}{2}} \tilde{\rho}_{n_1 n'_1}(\mathbf{s}) \tilde{\chi}_{n'_1 n'_2}(\mathbf{q}) \tilde{\rho}_{n'_2 n_2}(-\mathbf{s}) e^{i(l^*)^2 \mathbf{s} \times \mathbf{q}} \end{aligned} \quad (\text{D7})$$

Let us consider the direct and exchange terms separately. In the direct term, one  $n'$  index can always be summed freely, while the other is constrained by the Fermi occupation factor  $N_F$ . The sum over the free  $n'$  gives, according to Eqn. (D4)

$$\begin{aligned} \text{direct term} &= \sum_{n'_2} N_F(n'_2) \tilde{\rho}_{n_1 n_2}(\mathbf{q}) \langle n'_2 | e^{-i(\frac{1}{c}-c)\mathbf{q} \cdot \boldsymbol{\eta}} | n'_2 \rangle \\ &- \sum_{n'_1} N_F(n'_1) \tilde{\rho}_{n_1 n_2}(\mathbf{q}) \langle n'_1 | e^{-i(\frac{1}{c}-c)\mathbf{q} \cdot \boldsymbol{\eta}} | n'_1 \rangle \end{aligned} \quad (\text{D8})$$

The two terms are immediately seen to cancel. Now let us turn to the exchange terms, and consider the one that has the factor  $N_F(n'_2)$ , and a free sum over  $n'_1$ . In this term, one can use Eqn. (D6) to exchange the  $\tilde{\rho}$  and  $\tilde{\chi}$  matrix elements to obtain

$$\text{exchange term} = - \sum_{n'_2} N_F(n'_2) \sum_{n'_1} \tilde{\chi}_{n_1 n'_1}(\mathbf{q}) \times \int \frac{d^2 s}{(2\pi)^2} v(s) e^{-\frac{s^2 l^2}{2}} \tilde{\rho}_{n'_1 n'_2}(\mathbf{s}) \tilde{\rho}_{n'_2 n_2}(-\mathbf{s}) \quad (\text{D9})$$

Notice that the phase factor  $e^{i(l^*)^2 \mathbf{s} \times \mathbf{q}}$  has been cancelled by an opposite phase factor from Eqn. (D6). Now the angular  $\mathbf{s}$  integral forces  $n'_1 = n_2$  for a rotationally invariant potential, and the result contains the Fock energy of the state  $n_2$

$$\epsilon^F(n_2) \tilde{\chi}_{n_1 n_2}(\mathbf{q}) \quad (\text{D10})$$

Similarly, the other exchange term proportional to  $N_F(n'_1)$  ends up giving  $-\epsilon^F(n_1) \tilde{\chi}_{n_1 n_2}(\mathbf{q})$ . Due to the peculiar nature of the Hamiltonian the Hartree energy is a constant independent of the CF-LL index, and the difference of the Fock energies is the same as the difference of the full HF energies. Thus, the exchange contributions cancel the diagonal term  $(\epsilon(n_1) - \epsilon(n_2)) \tilde{\chi}_{n_1 n_2}(\mathbf{q})$ , and  $\tilde{\chi}_{n_1 n_2}(\mathbf{q})$  is indeed an left eigenvector with zero eigenvalue for  $\mathcal{H}$ . Also, this property is independent of the form of  $v(q)$  as long as it is rotationally invariant.

## APPENDIX E: Activation gaps

Now we need to find the energy cost of producing a widely separated particle-hole (PH) pair. This will be done by evaluating

$$\Delta_a = \langle \mathbf{p} + P | H | \mathbf{p} + P \rangle + \langle \mathbf{p} + H | H | \mathbf{p} + H \rangle - 2\langle \mathbf{p} | H | \mathbf{p} \rangle \quad (\text{E1})$$

$$= \int_q E(P) + E(H) \quad (\text{E2})$$

$$\int_q = \frac{1}{2} \int \frac{d^2 q}{4\pi^2} v(q) e^{-q^2 t^2 / 2}. \quad (\text{E3})$$

where  $P$  denotes a particle added to the state labeled  $\mu = (n = p, m = 0)$  and  $H$  denotes a state in which a hole has been made in the state  $\mu = (n = p - 1, m = 0)$ . Let us consider

$$E(P) = \langle \mathbf{p} | d_\mu d_1^\dagger d_2 d_3^\dagger d_4 d_\mu^\dagger | \mathbf{p} \rangle \rho_{12} \rho_{34} \quad (\text{E4})$$

In performing the contractions we

- Do not make any contractions within  $H$ . This gets rid of  $E_0 = \langle \mathbf{p} | H | \mathbf{p} \rangle$ , the ground state energy.
- Do not contract 1 with 2 or 3 with 4 since  $v(0) = 0$ .

We end up with

$$\int_q [\delta_{\mu 1} \delta_{23} \delta_{4\mu} (1 - n_1^F)(1 - n_2^F)(1 - n_4^F) - \delta_{\mu 3} \delta_{14} \delta_{2\mu} (1 - n_3^F)(n_4^F)(1 - n_2^F)] \rho_{12} \rho_{34}$$

Since  $4 = \mu = 1$  in the first term, we can drop  $(1 - n_1^F)(1 - n_4^F)$  and for similar reasons  $(1 - n_3^F)(1 - n_2^F)$  in the second giving us

$$E(P) = \sum_{m_2=0}^{\infty} \sum_{n_2=p}^{\infty} \rho_{\mu 2}(\mathbf{q}) \rho_{2\mu}(-\mathbf{q}) \quad (\text{E5})$$

$$- \sum_{m_2=0}^{\infty} \sum_{n_2=0}^{p-1} \rho_{2\mu}(\mathbf{q}) \rho_{\mu 2}(-\mathbf{q}). \quad (\text{E6})$$

Since the sum over  $m_2$  is unrestricted, we can use completeness and  $e^{-i\mathbf{q} \cdot \mathbf{R}} \cdot e^{-i\mathbf{q} \cdot \mathbf{R}} = I$  to get rid of the  $m$ -index altogether. Thus we end up with

$$E(P) = \left( \sum_{n=p}^{\infty} |\rho_{pn}|^2 - \sum_{n=0}^{p-1} |\rho_{pn}|^2 \right) \quad (\text{E7})$$

$$= \left[ \langle p | \rho(q) \rho(-q) | p \rangle - 2 \sum_{n=0}^{p-1} |\rho_{pn}|^2 \right] \quad (\text{E8})$$

A similar calculation for the hole state gives (upon dropping the ground state energy as usual)

$$E(H) = \left[ -\langle p-1 | \rho(q) \rho(-q) | p-1 \rangle + 2 \sum_{n=0}^{p-1} |\rho_{p-1,n}|^2 \right] \quad (\text{E9})$$

where

$$\langle n | \rho(q) \rho(-q) | n \rangle = \sum_{n'=o}^{\infty} |\rho(q)_{nn'}|^2. \quad (\text{E10})$$

Putting all the pieces together, we obtain the gap.

## APPENDIX F: Critical fields for magnetic transitions

We need to calculate

$$E(p-r, r) = \langle \mathbf{p} - \mathbf{r}, \mathbf{r} | H | \mathbf{p} - \mathbf{r}, \mathbf{r} \rangle \quad (\text{F1})$$

the energy in a state with  $p-r$  spin-up LL's and  $r$  spin-down LL's. Since the HF calculation for the spinless case is very similar, this treatment will be brief. We write

$$H = \sum_{1234} \int_q d_1^\dagger d_2 d_3^\dagger d_4 \rho_{12} \rho_{34} \quad (\text{F2})$$

with the understanding that a label like 1 stands for the triplet  $(n_1, m_1, s_1)$ ,  $s$  being the spin. The matrix elements  $\rho_{ij}$  are defined by

$$\begin{aligned} \rho_{12} &= \\ &\langle 1 | e^{-i\mathbf{q} \cdot \mathbf{R}} \left[ e^{-i\mathbf{q} \cdot \boldsymbol{\eta} c} - c^2 f e^{-i\mathbf{q} \cdot \boldsymbol{\eta} / c} \right] | 2 \rangle \\ &= \rho_{m_1 m_2} \otimes \rho_{n_1 n_2} \otimes \delta_{s_1 s_2} \end{aligned}$$

and as a result

$$\begin{aligned} E(p-r, r) &= \\ &\int_q \sum_{n_1 n_2 s} n_1^F(s) (1 - n_2^F(s)) |\rho_{n_1 n_2}|^2 \underbrace{\sum_m \langle m | I | m \rangle}_{=n/p} \end{aligned}$$

where we acknowledge the fact that the occupation factors  $n_1^F$  and  $n_2^F$  can depend on the spin. We have also used the fact that the sum over all values of  $m$  is the degeneracy of each CF-LL,  $n/p$ . Carrying out the sums over  $n_1$  and  $n_2$ , we obtain

$$\begin{aligned} E(p-r, r) &= \\ &\frac{n}{p} \int_q \left[ \sum_{n_1=0}^{p-r-1} \langle n_1 | \rho(q) \rho(-q) | n_1 \rangle - \sum_{n_1, n_2=0}^{p-r-1} |\rho_{n_1 n_2}|^2 \right. \\ &\quad \left. + \sum_{n_1=0}^{r-1} \langle n_1 | \rho(q) \rho(-q) | n_1 \rangle - \sum_{n_1, n_2=0}^{r-1} |\rho_{n_1 n_2}|^2 \right] \end{aligned}$$

It is now straightforward to compute the critical field for the transition  $|\mathbf{p} - \mathbf{r}, \mathbf{r}\rangle \rightarrow |\mathbf{p} - \mathbf{r} - \mathbf{1}, \mathbf{r} + \mathbf{1}\rangle$  by invoking

$$E(p-r, r) - E(p-r-1, r+1) = g \left[ \frac{e}{2m_e} \right] B^c \frac{n}{p} \quad (\text{F3})$$

## APPENDIX G: Acronyms and Symbols

This paper invokes many symbols and acronyms, not all of which have been standardized. For the convenience of the reader, a list is supplied in Table (VI).

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TABLE VI Frequently used acronyms

|      |                                |
|------|--------------------------------|
| CB   | composite bosons               |
| CF   | composite fermions             |
| CQFM | continuum quantum ferromagnet  |
| CS   | Chern-Simons                   |
| EHT  | Extended hamiltonian theory    |
| GMP  | Girvin, MacDonald and Platzman |
| HLR  | Halperin, Lee and Read         |
| HF   | Hartree-Fock                   |
| LL   | Landau level                   |
| LLL  | lowest Landau level            |
| RPA  | random phase approximation     |
| TDHF | time-dependent Hartree-Fock    |
| WC   | Wigner crystal                 |
| ZDS  | Zhang-Das Sarma                |

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| Symbol   | Significance  |
|--|---|
|  | <b>Widely used quantities</b>   |
| $\nu$  | $= p/(2ps + 1)$ = filling fraction  |
| $2s$   | Number of vortices of flux tubes attached                                     |
| $p$  | Number of CF LL's   |
| $\hat{\mathbf{z}}$                               | unit vector along $z$ -axis   |
| $c^2$  | $2ps/(2ps + 1)$   |
| $m_e$ ( $m$ )                                    | Electron mass in free space (in solid)  |
| $A^*$ or $B^*$                                   | Potential or field seen by CF $A^* = A/(2ps + 1)$                             |
| $\Pi$  | Velocity operator for CF, $\mathbf{p} + e\mathbf{A}^*$                        |
| $g$  | $g$ -factor of electron or CF, taken to be .44                                |
| $l$ ( $l^* = l/\sqrt{1 - c^2}$ )                 | electron (CF) magnetic length   |
| $\mathbf{R}_e$ or $\mathbf{R}_v$ or $\mathbf{R}$ | Electron, pseudovortex, or CF guiding center coordinate                       |
| $\boldsymbol{\eta}_e$ or $\boldsymbol{\eta}$     | Electron or CF cyclotron coordinate   |
| $\bar{H}$  | Hamiltonian in the LLL  |
| $\bar{\rho}(\mathbf{q})$                         | Electron density in the LLL   |
| $\bar{\chi}(\mathbf{q})$                         | Constraint  |
| $\bar{\rho}^p(\mathbf{q})$                       | $\bar{\rho}(\mathbf{q}) - c^2\bar{\chi}(\mathbf{q})$ (preferred charge)       |
|  | <b>Quantities related to gaps</b>   |
| $\Delta_{a,p}$                                   | Activation or polarization gap  |
| $m_{a,p}^{(2s)}$                                 | Defined by $\Delta_{a,p} = eB^*/(m_{a,p}^{(2s)})$                             |
| $\delta$   | $\Delta/(e^2/\varepsilon l)$  |
| $\lambda$  | Defined by $v_{ZDS}(q) = 2\pi e^2 e^{-ql\lambda}/q$                           |
|  | <b>Magnetic quantities</b>  |
| $S$  | Number of spin up minus down CFs  |
| $E(S)$   | Ground state energy density   |
| $\mathcal{E}_{\pm}(k)$                           | Hartree Fock energy for up/down spin at momentum $k$                          |
| $ \mathbf{p} - \mathbf{r}, \mathbf{r}\rangle$    | CF state with $p - r$ LL's spin up and $r$ down.                              |
|  | <b>Matrix elements</b>  |
| $\rho_{n_1 n_2}$                                 | single-particle matrix element of $\bar{\rho}^p$ between LL's $n_1$ and $n_2$ |
| $\tilde{\chi}_{nn'}$                             | $\langle n   \exp -i\mathbf{q} \cdot \boldsymbol{\eta}/c   n' \rangle$        |